

KOSZUL DUALITY FOR LOCALLY CONSTANT FACTORISATION ALGEBRAS

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ABSTRACT. Generalising Jacob Lurie’s idea on the relation between the Verdier duality and the iterated loop space theory, we study the Koszul duality for locally constant factorisation algebras. We formulate an analogue of Lurie’s “nonabelian Poincaré duality” theorem (which is closely related to earlier results of Graeme Segal, of Dusa McDuff, and of Paolo Salvatore) in a symmetric monoidal stable infinity category carefully, using John Francis’ notion of excision. Its proof depends on our study of the Koszul duality for E_n -algebras in [11]. As a consequence, we obtain a Verdier type equivalence for factorisation algebras by a Koszul duality construction.

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0. INTRODUCTION

This paper is continuation of [10, 11], and together with these papers, is based on the author's thesis [12]. Building on results from [10] on the foundations for the theory of locally constant factorisation algebras, we study here the Koszul duality for these algebras. For this, we use the results of [11] on the local case of the Koszul duality for E_n -algebras.

0.0. Koszul duality for factorisation algebras. Lurie has discovered what he calls the “nonabelian Poincaré duality” theorem [9]. (According to him, closely related results were earlier obtained by Segal [15], McDuff [13] and Salvatore [14].) Classically, the Poincaré duality theorem concerns a locally constant sheaf of abelian groups, or more generally of “stable” (in homotopical sense) objects such as chain complexes or spectra, on a manifold. Lurie's theorem states that a form of the theorem also holds with *unstable* coefficients, rather than stable or abelian coefficients. By “unstable coefficients”, we mean the coefficients in a locally constant sheaf of spaces. One formulation of the classical Poincaré duality theorem is that the compactly supported cohomology of the sheaf is a *homology* theory, namely, forms a cosheaf. Lurie's discovery is that the suitable homology theory for unstable coefficients is the topological chiral homology, which generalises the cosheaf homology. This homology theory determines a locally constant factorisation algebra, rather than a cosheaf. The following is a formulation of Lurie's theorem using the language of factorisation algebras (and sheaves).

Theorem 0.0 (Lurie). *Let M be a manifold of dimension n . Let E be a locally constant sheaf of based spaces on M . If every stalk of E has connectivity at least n , then the (locally constant) prealgebra E^+ of spaces on M defined by $E^+(U) = \Gamma_c(U, E)$, the compactly supported cohomology, is a factorisation algebra.*

He notes that the stalk of E^+ is the n -fold loop space of the stalk of E , and the structure of a (pre)algebra of E^+ globalises the E_n -algebra structure which characterises n -fold loop spaces in the iterated loop space theory. As a consequence, the theorem leads to a globalisation of the iterated loop space theory in the form of an equivalence between suitable infinity categories of sheaves and of factorisation algebras. This is an unstable counterpart of the Verdier duality theorem expressed as an equivalence between sheaves and cosheaves valued in a stable infinity category.

Iterated loop space theory is an instance of the Koszul duality [4] for E_n -algebras, and locally constant factorisation algebras globalise E_n -algebras. This motivates one to consider the Koszul duality for factorisation algebras, and look for a generalisation of the Poincaré and the Verdier theorems in this context.

For this purpose, we have defined *compactly supported* factorisation homology (Section 3). Given a locally constant factorisation algebra A on a manifold M , we denote the compactly supported homology on an open submanifold U by $\int_U^c A$. The association $A^+ : U \mapsto \int_U^c A$ is then a precoalgebra on M . The question then is how close A^+ is to a factorisation coalgebra.

Unfortunately, there arises a problem very soon. Namely, while topological chiral homology behaves very nicely in a symmetric monoidal infinity category in which the monoidal multiplication functor preserves sifted homotopy colimits variablewise (see [9, 10]), this assumption on sifted colimits is essential. Even though this condition is satisfied often in practice, if we would like to also consider factorisation coalgebras, then we would need the monoidal operations to also preserve sifted homotopy *limits*. This is a very strong constraint, even though it is satisfied in Lurie's context.

One of the principal aims of the present work is to remove this constraint in some other contexts which arise in practice, at least for the purpose of generalising Lurie’s results. In this direction, we have obtained quite satisfactory results by restricting our attention to algebras which are *complete* with respect to a suitable filtration. We have shown that many algebras which arise in practice are of this kind (after some natural procedure of completion, if necessary).

Indeed, we have established in [11] a very good theory of the Koszul duality in the local case of E_n -algebras, in a complete filtered context. In this paper, we show that the desired global results follow from this good local theory. Let us overview the results of [11] and then the main results of the present work.

0.1. Koszul duality for complete E_n -algebras. Let n be a finite non-negative integer.

Let us review the setting of [11]. Let \mathcal{A} be a symmetric monoidal infinity category. We assume that it has a filtration which is compatible with the symmetric monoidal structure in a suitable way. Primary examples are the category of filtered objects in a reasonable symmetric monoidal infinity category and a symmetric monoidal stable infinity category with a compatible t-structure [9] (satisfying a mild technical condition). Another family of examples is given by functor categories admitting the Goodwillie calculus [5], where the filtration is given by the degree of excisiveness. Indeed, we also assume that \mathcal{A} is stable in the sense stated in Lurie’s book [9]. (See Toën–Vezzosi’s [18] for the origin of the notion.) However, our monoidal structure is not the direct sum (which does not have the kind of compatibility with the filtration we need), but is like the (derived) tensor product of chain complexes, and the smash product of spectra, so our context is “nonabelian”. We further assume that \mathcal{A} is complete with respect to the filtration in a suitable sense. The mentioned examples admit completion, and in these examples, the category \mathcal{A} we indeed work in is the category of complete objects in the mentioned category, equipped with completed symmetric monoidal structure. These categories satisfy a few further technical assumptions we need, which we shall not state here.

In such a complete filtered infinity category \mathcal{A} , any algebra comes with a natural filtration with respect to which it is complete. In the mentioned examples, the towers associated to the filtration are the canonical (or “defining”) tower, the Postnikov tower, and the Taylor tower, and the objects we deal with are the limits of the towers. We have established the Koszul duality for E_n -algebras in \mathcal{A} which is *positively* filtered. The corresponding restriction on the filtration of coalgebras is given by another condition which we call **copositivity**. The theorem is as follows.

Theorem 0.1 ([11, Theorem 0.0]). *Let \mathcal{A} be as above. Then the constructions of Koszul duals give inverse equivalences*

$$\mathrm{Alg}_{E_n}(\mathcal{A})_+ \xleftarrow{\sim} \mathrm{Coalg}_{E_n}(\mathcal{A})_+$$

between the infinity category of positive augmented E_n -algebras and copositive augmented E_n -coalgebras in \mathcal{A} .

0.2. The Poincaré and the Verdier theorems. For obtaining global results from the local theory of [11], a breakthrough was the discovery by Francis of the notion of *excision* [2, 3]. Excision is concerned with what happens to the value of a prealgebra when a manifold is glued as in the composition in a cobordism category. Namely, let A be a prealgebra on a manifold M , and suppose an open submanifold U is cut into two pieces V and W along a codimension 1 submanifold N whose normal bundle is trivialised. Then one finds an E_1 -algebra, which we shall denote by $A(N)$, by restricting A to a tubular neighbourhood of N (diffeomorphic to $N \times \mathbb{R}^1$ by the trivialisation) and then pushing it down to \mathbb{R}^1 . One finds that $A(V)$ and $A(W)$ are

right and left modules respectively over $A(N)$, and the **excision** property requires that the canonical map

$$A(V) \otimes_{A(N)} A(W) \longrightarrow A(U)$$

be an equivalence in every such situation (where the tensor product should be understood as “derived”, if there is also an underived, i.e., not homotopy invariant, tensor product).

Francis proved that excision property characterises topological chiral homology, and have applied this theorem to give a simple proof of Lurie’s nonabelian Poincaré duality theorem [2, 3]. The excision property gives a convenient way to compute topological chiral homology.

Influenced by this work, we *formulate* the Poincaré duality theorem in our context using excision, and in this form, the theorem holds if the local theory is good enough. We say that an augmented locally constant factorisation algebra on an n -dimensional manifold is **positive** if it is locally so as an E_n -algebra.

Theorem 0.2 (Theorem 4.11). *Let A be a positive augmented locally constant factorisation algebra on M , valued in \mathcal{A} as in the previous section. Then the pre-coalgebra A^+ defined by $A^+(U) = \int_U^c A$, satisfies excision.*

In order to show the ubiquity of algebras to which this theorem applies, we have shown that *any* augmented factorisation algebra (taking values in a reasonable symmetric monoidal stable infinity category) comes with a canonical positive filtration (**Proposition 4.24**). It follows that the Poincaré theorem holds for its completion (Corollary 4.25).

See Section 4.4 for another example.

Remark 0.3. Related results can be found in the work of Francis [3], and Ayala and Francis [0].

Theorem 0.2 also leads to the Koszul duality for factorisation algebras, as a Verdier type equivalence of categories. Let $\mathrm{Alg}_M(\mathcal{A})_+$ (resp. $\mathrm{Coalg}_M(\mathcal{A})_+$) denote the infinity category of positive (resp. copositive) augmented prealgebras (resp. pre-coalgebras) on M which is locally constant in a suitable sense, and satisfies excision.

Theorem 0.4 (Theorem 4.15). *Let \mathcal{A} be as above. Then the functor*

$$(\)^+ : \mathrm{Alg}_M(\mathcal{A})_+ \longrightarrow \mathrm{Coalg}_M(\mathcal{A})_+$$

is an equivalence.

0.3. Outline. **Section 1** is for introducing conventions which are used throughout the main body.

In **Section 2**, we discuss excision.

In **Section 3**, we introduce compactly supported factorisation homology, and investigate its symmetric monoidal functoriality in manifolds.

In **Section 4**, we formulate the Poincaré duality for a factorisation algebra as a relation between the compactly supported factorisation homology and the Koszul dual of the factorisation algebra. We also investigate this for factorisation algebras in the situation opposite to Lurie’s. This is another situation where problem about sifted limits does not arise.

We then prove the Poincaré duality theorem for complete factorisation algebras. We compare the cases of (globally) constant algebras of this theorem, with an implication of the Morita theoretic functoriality of the Koszul duality [11, Theorem 4.22] on topological field theories. We also discuss particular algebras to which the theorem applies, including one which is of interest from quantum field theory in Costello–Gwilliam’s framework [1] (see **Theorem 4.31** for the result).

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1. TERMINOLOGY, NOTATIONS AND CONVENTIONS

1.0. Category theory. As in [10, 11], we adopt the following convention for the terminology.

By a **1-category**, we always mean an *infinity* 1-category. We often call a 1-category (namely an infinity 1-category) simply a **category**. A category with discrete sets of morphisms (namely, a “category” in the more traditional sense) will be called (1, 1)-category, or a *discrete* category.

In fact, all categorical and algebraic terms will be used in *infinity* (1-) categorical sense without further notice. Namely, categorical terms are used in the sense enriched in the *infinity* 1-category of spaces, or equivalently, of infinity groupoids, and algebraic terms are used freely in the sense generalised in accordance with the enriched categorical structures.

For example, for an integer n , by an n -category (resp. *infinity* category), we mean an *infinity* n -category (resp. infinity infinity category). We also consider multicategories. By default, multimaps in our multicategories will form a *space* with all higher homotopies allowed. Namely, our “*multicategories*” are “infinity operads” in the terminology of Lurie’s book [9].

Remark 1.0. We usually treat a space relatively to the structure of the standard (infinity) 1-category of spaces. Namely, a “*space*” for us is usually no more than an object of this category. Without loss of information, we shall freely identify a space in this sense with its fundamental infinity groupoid, and call it also a “*groupoid*”. Exceptions in which the term “space” means not necessarily this, include a “Euclidean space”, the “total space” of a fibre bundle, etc., in accordance with the common customs.

The following notations and terminology will be used as in [10, 11].

We use the following notations for over and under categories. Namely, if \mathcal{C} is a category and x is an object of \mathcal{C} , then we denote the category of objects \mathcal{C} lying over x , i.e., equipped with a map to x , by $\mathcal{C}_{/x}$. We denote the under category for x , in other words, $((\mathcal{C}^{\text{op}})_{/x})^{\text{op}}$, by $\mathcal{C}_{x/}$.

More generally, if a category \mathcal{D} is equipped with a functor to \mathcal{C} , then we define $\mathcal{D}_{/x} := \mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{/x}$, and similarly for $\mathcal{D}_{x/}$. Note here that $\mathcal{C}_{/x}$ is mapping to \mathcal{C} by the functor which forgets the structure map to x . Note that the notation is abusive in that the name of the functor $\mathcal{D} \rightarrow \mathcal{C}$ is dropped from it. In order to avoid this abuse from causing any confusion, we shall use this notation only when the functor $\mathcal{D} \rightarrow \mathcal{C}$ that we are considering is clear from the context.

By the **lax colimit** of a diagram of categories indexed by a category \mathcal{C} , we mean the Grothendieck construction. We choose the variance of the laxness so the lax

colimit projects to \mathcal{C} , to make it an op-fibration over \mathcal{C} , rather than a fibration over \mathcal{C}^{op} . (In particular, if $\mathcal{C} = \mathcal{D}^{\text{op}}$, so the functor is contravariant on \mathcal{D} , then the familiar fibred category over \mathcal{D} is the *op*-lax colimit over \mathcal{C} for us.) Of course, we can choose the variance for lax *limits* compatibly with this, so our lax colimit generalises to that in any 2-category.

1.1. Symmetric monoidal structure. The following explicit definition of a symmetric monoidal category will be used. Namely, we follow Toën [17] to define a symmetric monoidal (infinity 1-) category as an infinity 1-categorical generalisation of Segal’s Γ -category [16], in accordance with Lurie’s book [9, Definitions 2.0.0.7, 2.1.3.7]. Namely, let Fin_* denote the category of pointed finite sets (equivalently, the opposite of Segal’s category Γ). For a finite set S , denote by $S_+ = S \amalg \{*\}$ the object of Fin_* obtained by externally adding a base point “ $*$ ” to S .

Definition 1.1. Let Cat denote the (2-)category of categories (with some fixed limit for their sizes). Let \mathcal{C} be a category, i.e., an object of Cat . Then a **pre- Γ -structure** on \mathcal{C} consists of a functor $\mathcal{A}: \text{Fin}_* \rightarrow \text{Cat}$ together with an equivalence $\mathcal{C} \simeq \mathcal{A}(\mathbb{S}^0)$, where \mathbb{S}^0 denotes the two pointed set with one base point.

A pre- Γ -structure as above is a **symmetric monoidal** structure if for every finite set S (including the case $S = \emptyset$), Segal’s map

$$(1.2) \quad \mathcal{A}(S_+) \longrightarrow \mathcal{A}(\mathbb{S}^0)^S$$

is an equivalence [16, Definition 2.1].

A **pre- Γ -category** is a category equipped with a pre- Γ -structure, or equivalently, just a functor $\text{Fin}_* \rightarrow \text{Cat}$. It is a **symmetric monoidal** category if the pre- Γ -structure is in fact a symmetric monoidal structure.

The category of *maps* (“**symmetric monoidal functors**”) between symmetric monoidal categories is by definition, the category of maps of the functors on Fin_* .

Let \mathcal{A} be a symmetric monoidal category (i.e., a pre- Γ -category $\text{Fin}_* \rightarrow \text{Cat}$ satisfying the required condition). Then through the equivalence (1.2), the map

$$(1.3) \quad \mathcal{A}(S_+) \longrightarrow \mathcal{A}(\mathbb{S}^0)$$

induced from the map which collapses S to the (non-base) point can be considered as a functor

$$(1.4) \quad \mathcal{C}^S \longrightarrow \mathcal{C},$$

where \mathcal{C} is the underlying category $\mathcal{A}(\mathbb{S}^0)$. These can be considered as “multiplication” operations on \mathcal{C} which results from the symmetric monoidal structure. In fact, since $\mathcal{A}(S_+)$ can be replaced by \mathcal{C}^S , so Segal’s maps will be the identities, a symmetric monoidal structure on \mathcal{C} amounts to the operations (1.4) together with suitable compatibility data among them.

In our notation, we often use the same symbol for a symmetric monoidal category and its underlying category, when this seems to cause no confusion. On the other hand, the name for a symmetric monoidal structure will often be something like “ \otimes ”, in which case the name of the multiplication operation (1.4) will be \otimes_S . (If the operations already have names such as \otimes_S , we will name the symmetric monoidal structure after them, so the stated rule will apply.)

We shall also need to consider *partially defined* monoidal structures. The following simple definition will suffice for our purposes.

Definition 1.5. A **partial** symmetric monoidal category is a pre- Γ -category \mathcal{A} for which Segal’s maps (1.2) are fully faithful functors.

The category of *maps* (“**symmetric monoidal functors**”) of partial symmetric monoidal categories is by definition, the category of maps of the functors on Fin_* .

In a partial symmetric monoidal category \mathcal{A} , (1.3) is a multiplication operation defined only on the full subcategory $\mathcal{A}(S_+)$ of \mathcal{A}^S . We shall often denote $\mathcal{A}(S_+)$ by $\mathcal{A}^{(S)}$, so Segal's map will be $\mathcal{A}^{(S)} \hookrightarrow \mathcal{A}^S$, while the multiplication will be $\mathcal{A}^{(S)} \rightarrow \mathcal{A}$.

1.2. Manifolds and factorisation algebras. In this paper, every **manifold without boundary** is assumed to be the interior of a *specified* smooth compact manifold with (possibly empty) boundary. Namely, such a manifold U comes *equipped* with a smooth compact closure which will be usually denote by \overline{U} . By an **open embedding** $U \hookrightarrow V$ of such manifolds, where U and V are specified as the interior of compact \overline{U} and \overline{V} respectively, we mean an open embedding $U \hookrightarrow V$ in the usual sense which extends to a smooth immersion $\overline{U} \rightarrow \overline{V}$. By definition an **open submanifold** is a manifold embedded in this sense.

There will be a switch in notations from [10] accordingly. Firstly, for a manifold M without boundary, $\text{Open}(M)$ introduced in [10, Section 2.0] will now denote the open submanifolds in the above sense. Note that by [10, Corollary 2.39, Example 2.40], this class of manifolds are sufficient for understanding locally constant factorisation algebras. From the standpoint of the original conventions, this means that we work only with prealgebras which are left Kan extensions of their restriction to this class of manifolds (but the category of locally constant factorisation algebras remains unchanged).

Similarly, in this paper, $\text{Disk}(M)$ and $\text{Disj}(M)$ used in [10, Section 2.0] following Lurie [9], contain as objects (disjoint unions of) disks which are diffeomorphic to the interior of the standard closed disk, and is embedded in M in the above sense. All results of [10] are valid under these switched notations.

In this paper, we assume as in [10] that the target category \mathcal{A} of prealgebras has sifted colimits, and the monoidal multiplication functor on \mathcal{A} preserves sifted colimits variable-wise. Equivalently, the monoidal multiplication should preserve sifted colimits for all the variables at the same time.

By a (symmetric) **monoidal** structure on a *stable* category, we mean a (symmetric) monoidal structure on the underlying category for which the monoidal multiplication functors are *exact* in each variable. Note that this and the above implies that when we consider a symmetric monoidal stable category \mathcal{A} for the target of prealgebras, \mathcal{A} is closed under all colimits, and the monoidal multiplication functors preserve all colimits variable-wise.

For all other notations and terminology about factorisation algebras, we follow [10].

2. EXCISION PROPERTY OF A FACTORISATION ALGEBRA

2.0. Constructible algebra on a closed interval. Let I be a closed interval. Let $\text{Open}(I)$ be the poset of open subsets of I . This has a partially defined symmetric monoidal structure given by taking disjoint union. A **prealgebra** on I is defined to be a symmetric monoidal functor on $\text{Open}(I)$.

As a manifold with boundary, I has a natural stratification given by $\partial I \subset I$. We shall define the class of prealgebras on I which we shall call *constructible* factorisation algebras, where the constructibility is with respect to the mentioned stratification of I .

Let $\text{Disk}(I)$ be the full subcategory of the poset $\text{Open}(I)$ consisting of objects of $\text{Disk}(I - \partial I)$ and collars of either point of ∂I . This is a symmetric multicategory by inclusion of disjoint unions.

Let us say that a functor defined on the underlying poset (of “colours”) of $\text{Disk}(I)$ is **constructible** if it inverts morphisms from $\text{Disk}(I - \partial I)$ and morphisms between collars of points of ∂I .

Definition 2.0. Let \mathcal{A} be a symmetric monoidal category.

Then a prealgebra on I in \mathcal{A} is said to be **constructible** (with respect to the stratification of I as a manifold with boundary) if its restriction to $\text{Disk}(I)$ is constructible. Let us denote by $\text{PreAlg}_I(\mathcal{A})$ the category of *constructible* prealgebras on I .

A **constructible factorisation algebra** (or just “**constructible algebra**”) on I is an algebra on $\text{Disk}(I)$ whose underlying functor on colours is constructible. The category of constructible algebras on I in \mathcal{A} will be denoted by $\text{Alg}_I(\mathcal{A})$.

In fact, we can identify the category of constructible algebra on I with a right localisation of the category of constructible prealgebras consisting of those prealgebras whose underlying functor is a left Kan extension from a certain full subcategory, denoted by $\text{Disj}(I)$, of $\text{Open}(I)$. Namely, this full subcategory $\text{Disj}(I)$ is the smallest full subcategory of $\text{Open}(I)$ containing $\text{Disk}(I)$, and is closed under the partial monoidal structure of $\text{Open}(I)$, the disjoint union operation.

In fact, the only interesting open submanifold of I is I itself, and we have the following.

Lemma 2.1. *Let A be a constructible prealgebra on I . Then A is a (constructible) factorisation algebra if and only if the map $\text{colim}_{\text{Disj}(I)} A \rightarrow A(I)$ is an equivalence.*

We will next see that this colimit can be calculated as a tensor product in the following way. Given a constructible (pre-)algebra on I , note that the values of A on any object of $\text{Disk}(I - \partial I)$ are canonically equivalent to each other (since they are all canonically equivalent to $A(I - \partial I)$). Similarly, the values of A on any collar of left end point of I is canonically equivalent to each other (since these collars are totally ordered), and similarly around the right end point. Denote these objects by B, K, L respectively. Then we want to see in particular, that there functorially exists a structure of associative algebra on B , a structure of its right module on K , a structure of its left module on L for which there is a natural map $K \otimes_B L \rightarrow A(I)$.

Remark 2.2. Moreover, there will be a natural right B -module map $B \rightarrow K$, and a left B -module map $B \rightarrow L$. Naturally, this can be understood as that K (resp. L) is an E_0 -algebra in the category of right (resp. left) B -modules. (This is a factorisation algebra on a point which associates the object B to the empty set.)

There is an obvious way to modify the definition of a constructible prealgebra so that these extra structures will not come with the structure of A .

In order to do this, we extend isotopy invariance result from [10, Section 2.3] to the present (actually very simple) context. Let \mathbb{E}_I denote the multicategory which has the same objects as $\text{Disk}(I)$, but the space of multimaps $\{U_i\}_i \rightarrow V$ in \mathbb{E}_I is the space formed by pairs consisting of an embedding $f: \coprod_i U_i \hookrightarrow V$ and an isotopy of each U_i in I from the defining inclusion $U_i \hookrightarrow I$ to $f|_{U_i}$.

For distinction between a multicategory and its underlying category (of “colours”), let us denote by $\mathbb{E}_{1,I}$ the underlying category of \mathbb{E}_I . Then $\mathbb{E}_{1,I}$ is equivalent to the poset of subsets of ∂I consisting of *at most* one element.

There is a morphism $\text{Disk}(I) \rightarrow \mathbb{E}_I$ of multicategories, and clearly, the underlying functor $\text{Disk}_1(I) \rightarrow \mathbb{E}_{1,I}$, where we have put the subscript 1 for distinction, is a localisation inverting inclusions in $\text{Disk}_1(I - \partial I)$ and inclusions of collars of a point of ∂I (namely, those morphisms which are required to be inverted by a constructible prealgebra). Note that this is how we found the objects B, K, L above.

In particular, if A is a prealgebra on I , then its underlying functor extends to a functor on $\mathbb{E}_{1,I}$ if and only if A is constructible. Moreover, the extension is unique.

\mathbb{E}_I is slightly more involved than $\mathbb{E}_{1,I}$, but it is still homotopically discrete, and here is a complete description of it: Given a functor on $\mathbb{E}_{I,1}$, a structure on it of an algebra on \mathbb{E}_I , is exactly a structure of associative algebra on B , a structure of right B -module on K , a structure of left B -module map on the map $B \rightarrow K$, a structure of a left B -module on L , a structure of a left B -module map on the map $B \rightarrow L$.

With this description available, a direct inspection shows that the restriction through the morphism $\text{Disk}(I) \rightarrow \mathbb{E}_I$ induces an equivalence between the category of constructible algebra on I , and the category of algebras over \mathbb{E}_I . (For example, we could consider as an intermediate step, a symmetric multicategory defined similarly to \mathbb{E}_I , but using only isotopies (and their isotopies etc.) which are piecewise linear.)

Let us now introduce a similarly ‘localised’ version of $\text{Disj}(I)$. Namely we define a suitable variant of Lurie’s $D(M)$ [9].

Let Man^1 be the following category. Namely, its object is a 1-dimensional manifold with boundary which is a finite disjoint union (coproduct) of open or half-open intervals. The space of morphisms is the space of embeddings each of which sends any boundary point to a boundary point.

Define $D(I) := \text{Man}^1_I$. Its objects are open submanifolds of I which are homeomorphic to a finite disjoint union of disks. The space of maps $U \rightarrow V$ is the space formed by embeddings $f: U \hookrightarrow V$ together with an isotopy from the defining inclusion $U \hookrightarrow I$ to $f: U \hookrightarrow I$.

Lemma 2.3. *The functor $\text{Disj}(I) \rightarrow D(I)$ is cofinal.*

Proof. Proof is similar to the proof of [9, Proposition 5.3.2.13 (1)].

We apply Joyal’s generalisation of Quillen’s theorem A [7].

Let $D \in \text{Man}^1$ with an embedding $i: D \hookrightarrow I$ be defining an object of $D(I)$. (Denote the object simply by D .) Then we want to prove that the category $\text{Disj}(I)_{D/}$ has contractible classifying space.

In other words, we want to prove that the category

$$\text{laxcolim}_{E \in \text{Disj}(I)} \text{Fibre} [\text{Emb}(D, E) \longrightarrow \text{Emb}(D, I)],$$

fibre taken over i , has contractible classifying space, which is the colimit of the same diagram (rather than the lax colimit in the 2-category of categories), and thus equivalent to

$$\text{Fibre} \left[\text{colim}_{E \in \text{Disj}(I)} \text{Emb}(D, E) \longrightarrow \text{Emb}(D, I) \right].$$

In this last step, we have used the standard equivalence between the category of spaces over $\text{Emb}(D, I)$, and the category of local systems of spaces on $\text{Emb}(D, I)$ (to be elaborated on in Remark below for completeness).

In fact, we can prove that the map $\text{colim}_{E \in \text{Disj}(I)} \text{Emb}(D, E) \rightarrow \text{Emb}(D, I)$ is an equivalence as follows.

Let D_0 be the union of the components of D which are open intervals. Choose a homeomorphism $D_0 \simeq S \times \mathbb{R}^1$ for a finite set S . In particular, we have picked a point in each component of D_0 , corresponding to the origin in \mathbb{R}^1 , together with the germ of a chart at the chosen points. Then, given an embedding $D_0 \hookrightarrow U := E - \partial E$, where $E \in \text{Disj}(I)$, restriction of it to the germs of charts at the chosen points gives us an injection $S \hookrightarrow U$ together with germs of charts in U at the image of S . This defines a homotopy equivalence of $\text{Emb}(D_0, U) (\simeq \text{Emb}(D, E))$ with the space of germs of charts around distinct points in U , labeled by S .

Furthermore, this space is fibred over the configuration space $\text{Conf}(S, U) := \text{Emb}(S, U) / \text{Aut}(S)$, with fibres equivalent to $\text{Germ}_0(\mathbb{R}^1) \wr \text{Aut}(S)$, where $\text{Germ}_0(\mathbb{R}^1)$ is from [9, Notation 5.2.1.9].

It follows that the task has been reduced to proving that the map

$$\text{colim}_{E \in \text{Disj}(I)} \text{Conf}(\pi_0(D_0), E - \partial E) \longrightarrow \text{Conf}(\pi_0(D_0), I - \partial I)$$

is an equivalence.

However, for any finite set S , the cover determined by the functor $E \mapsto \text{Conf}(S, E - \partial E) \subset \text{Conf}(S, I - \partial I)$ satisfies the hypothesis for the generalised Seifert–van Kampen theorem [9]. \square

The following is a side remark on a result we have used.

Remark 2.4. In the proof, we have used an equivalence between the category of spaces over a space X , and the category of local systems of spaces on X . This is simply a special case of a standard fact on Grothendieck fibrations. Indeed, spaces are also known as groupoids, so “a space over X ” is a rephrasing of “a category fibred over X in groupoids” (both consist of identical data with equivalent required properties), and every functor over X preserves Cartesian maps. (Note that every functor with target a groupoid is a fibration, and a map in the source is Cartesian if and only if it is an equivalence.)

Let us denote by $\text{D}(I)_{\partial I}$ the full subcategory of $\text{D}(I)$ consisting of objects each of which contains (as a submanifold of I) both points of ∂I . Note that the inclusion $\text{D}(I)_{\partial I} \rightarrow \text{D}(I)$ is cofinal.

Lemma 2.5. *The functor $\text{D}(I)_{\partial I} \rightarrow \Delta^{\text{op}}$ given by $U \mapsto \pi_0(I - U)$ (with the order inherited from the order in I) is an equivalence.*

Proof. A simple verification. \square

Let us denote by $\text{Disj}(I)_{\partial I}$ the similar full subposet of $\text{Disj}(I)$. In other words, $\text{Disj}(I)_{\partial I} := \text{D}(I)_{\partial I} \times_{\text{D}(I)} \text{Disj}(I)$.

Corollary 2.6. *The canonical functor $\text{Disj}(I)_{\partial I} \rightarrow \text{D}(I)_{\partial I}$ is cofinal.*

Proof. This follows from Lemma 2.3. Indeed, for every $U \in \text{D}(I)_{\partial I}$, the induced functor $(\text{Disj}(I)_{\partial I})_{U/} \rightarrow \text{Disj}(I)_{U/}$ is an equivalence. \square

Lemma 2.7. *The inclusion functor $\text{Disj}(I)_{\partial I} \hookrightarrow \text{Disj}(I)$ is cofinal.*

Proof. We apply Joyal’s generalisation of Quillen’s theorem A [7].

Namely, we would like to prove for any given $U \in \text{Disj}(I)$ that the category $(\text{Disj}(I)_{\partial I})_{U/}$ has contractible classifying space.

Let $(\text{Disj}(I)_{\partial I})_U$ denote the full subposet of $(\text{Disj}(I)_{\partial I})_{U/}$ consisting of maps $U \hookrightarrow V$ in $\text{Disj}(I)$ such that V is of the form $U \sqcup D$ for a collar D of $\partial I \cap (I - U)$ in I which is disjoint with U . Then the inclusion $(\text{Disj}(I)_{\partial I})_U \hookrightarrow (\text{Disj}(I)_{\partial I})_{U/}$ has a right adjoint, so induces a homotopy equivalence on the classifying spaces. However, the classifying space of $(\text{Disj}(I)_{\partial I})_U$ is contractible since it has maximum. \square

We conclude from Lemma 2.7, Corollary 2.6 and Lemma 2.5, that we have obtained a pair of cofinal functors

$$\text{Disj}(I) \longleftarrow \text{Disj}(I)_{\partial I} \longrightarrow \Delta^{\text{op}}.$$

Now let A be a constructible prealgebra on I , and let B, K, L be the algebras and modules obtained from A . Then it follows from the above constructions that the functor $A|_{\text{Disj}(I)_{\partial I}}$ is equivalent to the restriction to $\text{Disj}(I)_{\partial I}$ of the simplicial bar construction $B_{\bullet}(K, B, L): \Delta^{\text{op}} \rightarrow \mathcal{A}$.

In particular, we obtain a canonical equivalences

$$\operatorname{colim}_{\operatorname{Disj}(I)} A \xleftarrow{\sim} \operatorname{colim}_{\operatorname{Disj}(I)_{\partial I}} A \xrightarrow{\sim} \operatorname{colim}_{\Delta^{\operatorname{op}}} B_{\bullet}(K, B, L) = K \otimes_B L.$$

Let us denote the points of ∂I by x_+ and x_- . Recall that the underlying object of the algebra “ B ” is identified with $A(I - \partial I)$. Let us assume that our conventions identify $A(I - x_-)$ with the right B -module “ K ”, and $A(I - x_+)$ with the left B -module “ L ”.

In this way, objects $A(I - \partial I)$ and $A(I - x_{\pm})$ get a canonical structure of an algebra and its left/right module respectively.

Proposition 2.8. *Let A be a constructible prealgebra on I , and assume that, as a functor, A preserves filtered colimits. Then A is a factorisation algebra if and only if the canonical map*

$$A(I - x_-) \otimes_{A(I - \partial I)} A(I - x_+) \longrightarrow A(I)$$

is an equivalence. Here, the tensor product is with respect to the canonical structures.

2.1. Excision. Following Francis, we shall introduce the notion of excision and review after him, the relation between excision and other descent properties.

Definition 2.9. Let M be a manifold (without boundary). Then we say that a map $p: M \rightarrow I$ is **constructible** if $p: p^{-1}(I - \partial I) \rightarrow I - \partial I$ is locally trivial, i.e., is the projection of a fibre bundle.

Let $N := p^{-1}(t) \subset M$ be the fibre of a point $t \in I - \partial I$. Then N is a smooth submanifold of M of codimension 1, and its normal bundle in M is a trivial line bundle.

We can write $I = I_0 \cup_t I_1$ where I_0 is the points of I below or equal to t , and I_1 is the points of I above or equal to t . Accordingly, the total space M can be written in the glued form $M_0 \cup_N M_1$ where $M_i = p^{-1}I_i$.

Conversely, if M is given a decomposition $M_0 \cup_N M_1$ with N a submanifold of codimension 1 with trivial normal bundle, then we have a constructible map $p: M \rightarrow I$ for an interval I so the decomposition of M can be reconstructed as above from p . It suffices to choose a trivialisation of the normal bundle of N (in the desired orientation) to construct p .

The excision property is concerned with what happens to the value associated by a prealgebra when M is constructed by gluing as above.

Let $p: M \rightarrow I$ be constructible, then for a locally constant algebra A on M , it is immediate from the isotopy invariance that the prealgebra p_*A on I is constructible.

Definition 2.10. Let A be a locally constant prealgebra on M . We say that A satisfies **excision with respect to** a constructible map $p: M \rightarrow I$ if the prealgebra p_*A on I is a constructible factorisation algebra on I .

We say that A satisfies **excision** if for every $U \subset M$ equipped with a constructible map $p: U \rightarrow I$, $A|_U$ satisfies excision with respect to p .

The following is a formulation in our setting of a fundamental fact discovered by Francis, with proof also following his ideas.

Theorem 2.11 (Francis). *A locally constant prealgebra on a manifold is a factorisation algebra, namely its underlying functor is a left Kan extension from disjoint unions of disks [10, Section 2], if and only if it satisfies excision.*

In order to prove this, we recall the following definition and a theorem.

Definition 2.12 ([10, Definition 2.5]). Let \mathcal{C} be a category and let $\chi: \mathcal{C} \rightarrow \text{Open}(M)$ be a functor. For $i \in \mathcal{C}$, denote $\chi(i)$ also by U_i within this definition. We shall call this data a **factorising cover** which is **nice in Lurie's sense**, or briefly, **factorising l-nice cover**, of M if for any non-empty finite subset $x \subset M$, the full subcategory $\mathcal{C}_x := \{i \in \mathcal{C} \mid x \subset U_i\}$ of \mathcal{C} has contractible classifying space.

Theorem 2.13 ([10, Theorem 2.11]). *Let A be a locally constant algebra on M (in a symmetric monoidal category \mathcal{A} satisfying our assumption stated in Section 1.2). Then for any factorising l-nice cover determined by $\chi: \mathcal{C} \rightarrow \text{Open}(M)$, the map $A(M) \leftarrow \text{colim}_{\mathcal{C}} A\chi$ is an equivalence.*

Proof of Theorem 2.11. Let A be a locally constant factorisation algebra on a manifold M , and let us prove that A satisfies excision. For this purpose, let U be an open submanifold equipped with a constructible map $p: U \rightarrow I$. We need to prove that the constructible algebra $p_*(A|_U)$ on I is a left Kan extension from disjoint unions of subintervals.

For notational convenience, denote $A|_U$, which satisfies the same assumption as A does, just by A .

We want to prove that for every open submanifold $V \subset I$, the value $(p_*A)(V)$ is equivalent to $\text{colim}_{D \in \text{Disj}(V)} (p_*A)(D)$ by the canonical map. Namely, $A(p^{-1}V) = \text{colim}_{D \in \text{Disj}(V)} A(p^{-1}D)$.

Since the objects of $\text{Disj}(V)$ form a factorising l-nice cover of V , the functor $p^{-1}: \text{Disj}(I) \rightarrow \text{Open}(p^{-1}V)$ determines a factorising l-nice cover of $p^{-1}V$. Therefore, the result follows from Theorem 2.13.

The converse now follows as follows.

Firstly, if a locally constant prealgebra satisfies excision, then this prealgebra as well as the factorisation algebra obtained from it as a left Kan extension of its restriction to disjoint unions of disks, both satisfy excision. Then since every open submanifold of M (or rather its compact closure) has a handle body decomposition, two prealgebras coincide as soon as they coincide on open submanifolds diffeomorphic to a disk or $D^i \times \partial \overline{D}^j$.

However, the two prealgebras do coincide on disks by construction, and then also on $D^i \times \partial \overline{D}^j$ by inductively (on j) applying excision. \square

Remark 2.14. The construction similar to that in [10, Section 3.0] of the functoriality for the push-forward operation on the groupoid of locally trivial maps shows that the push-forward p_*A of a locally constant algebra is naturally functorial in p (on the groupoid of constructible maps).

3. COMPACTLY SUPPORTED FACTORISATION HOMOLOGY

Let M be a manifold without boundary, and let A be a locally constant factorisation algebra on M . Recall that A determines a functor $U \mapsto A(U) := \int_U A$ by factorisation homology. When A is equipped with an *augmentation*, namely, an algebra map $A \rightarrow \mathbf{1}$, we shall define *compactly supported* factorisation homology $\int_U^c A$ with coefficients in A , and shall make it into a symmetric monoidal *contravariant* functor of U .

Remark 3.0. Although this is implicit in our notation $\int_U^c A$, compactly supported homology will be defined as *dependent* on the compactification of U which comes with U in our convention (see Section 1.2).

For the purpose of definitions in this section, by an **interval**, we mean an oriented smooth compact connected manifold of dimension 1, with exactly one incoming boundary point and exactly one outgoing point with respect to the orientation.

3.0. Description of the object. Let us first define, for each open submanifold $U \subset M$, the compactly supported homology object $\int_U^c A$.

Let I be an interval with incoming point s and outgoing point t . We choose our conventions so a constructible algebra on I with stratification specified by its boundary, is given by an associative algebra B to be on the interior, a *right* B -module K to be on the point s , a *left* B -module L to be on the point t , and (right or left) B -module maps $B \rightarrow K$ and $B \rightarrow L$. Recall then the factorisation homology over I is $K \otimes_B L$ (Proposition 2.8).

Let \overline{U} denote the specified compact closure of U . For the construction, we choose a (constructible) C^∞ -map $p: \overline{U} \rightarrow I$ which is locally trivial over $I - \{s\}$, and such that $p^{-1}(t) = \partial\overline{U}$, so p restricts to a constructible map $U \rightarrow I - \{t\}$. By a constructible (resp. locally trivial) map, we always mean a smooth map which is constructible (resp. locally trivial) in the category of smooth manifolds and C^∞ maps.

With these data specified, we have an algebra p_*A on I . Let j denote the inclusion $I - \{t\} \hookrightarrow I$. Then the augmentation of A , and hence of p_*A , allows one to extend j^*p_*A (the restriction of p_*A along j) along j by putting the module $\mathbf{1}$ on t . Let us denote this augmented algebra on I by $j_! j^* p_* A$.

We then define the **compactly supported** factorisation homology over U to be

$$\int_U^c A := \int_I j_! j^* p_* A.$$

In other words, it is $A(U) \otimes_{A(\partial\overline{U})} \mathbf{1}$, where $A(\partial\overline{U})$ is the associative algebra we find on the interior of I from p_*A . Since the pushforward of the augmented algebra A is functorial in p on the groupoid of constructible maps by Remark 2.14, $\int_U^c A$ is unambiguously defined by A and U .

3.1. Functoriality. Next we would like to make the association $U \mapsto \int_U^c A$ contravariantly functorial in U . Let us start with some preparation. Let N_\bullet denote the nerve functor from the category of categories to the category of simplicial spaces. Then, since N_\bullet is fully faithful by [9, Proposition A.7.10], a functor $\text{Open}(M)^{\text{op}} \rightarrow \mathcal{A}$ is the same as a map $N_\bullet \text{Open}(M)^{\text{op}} \rightarrow N_\bullet \mathcal{A}$ of simplicial spaces.

Let Δ denote the category of combinatorial simplices. Its objects are non-empty totally ordered finite sets, and maps are order preserving maps. Then the data of a simplicial space is equivalent to the category fibred over Δ in groupoids, obtained by taking the lax colimit over Δ^{op} . Moreover, the desired simplicial map is then equivalent to a functor $\text{laxcolim}_{\Delta^{\text{op}}} N_\bullet \text{Open}(M)^{\text{op}} \rightarrow \text{laxcolim}_{\Delta^{\text{op}}} N_\bullet \mathcal{A}$ over Δ . Indeed, every map in a category fibred in groupoids is Cartesian over the base.

Let us denote $\text{laxcolim}_{\Delta^{\text{op}}} N_\bullet \text{Open}(M)^{\text{op}}$ by \mathcal{X} . Again, it suffices to construct a map $N_\bullet \mathcal{X} \rightarrow N_\bullet \text{laxcolim}_{\Delta^{\text{op}}} N_\bullet \mathcal{A}$ of simplicial spaces over $N_\bullet \Delta$. In order to construct this, we replace $N_\bullet \mathcal{X}$ by a simplicial space X_\bullet equivalent to it. Namely, we shall construct X_\bullet , an equivalence $X_\bullet \xrightarrow{\sim} N_\bullet \mathcal{X}$, and a map $X_\bullet \rightarrow N_\bullet \text{laxcolim}_{\Delta^{\text{op}}} N_\bullet \mathcal{A}$.

Let us first describe X_0 . We define it as the colimit (coproduct) over $N_0 \mathcal{X} = \text{colim}_{[k] \in N_0 \Delta} N_k \text{Open}(M)$ of certain spaces, each of which will turn out to be contractible. Namely, given an integer k and a k -nerve $U: U_0 \hookrightarrow \cdots \hookrightarrow U_k$ of $N_k \text{Open}(M)$, we associate to it the natural space X_U formed by pairs (p, I') , where

- (0) $p = (p_i)_{i \in [k]}$, where $p_i: \overline{U}_i \rightarrow I_i$ is a (C^∞) constructible map (in the C^∞ sense of the word) to an interval as before, and
- (1) $I' = (I'_i)_{i \in [k]}$, where I'_i is a subinterval of I_i

which are required to satisfy the following condition. Namely, give I_i a total order which recovers its topology, and for which $s_i < t_i$ for the incoming end point s_i and

the outgoing end point t_i , so $I_i = [s_i, t_i]$. Write $I'_i = [s'_i, t'_i]$ in this order. Then the required condition will be that

$$p_i^{-1}[s_i, t'_i] \subset p_j^{-1}[s_j, s'_j]$$

in U_j , whenever $i \leq j$.

We define X_0 to be the coproduct of X_U over all k and U .

Claim 3.1. *The map $X_0 \rightarrow N_0\mathcal{X}$ forgetting p and I' is an equivalence.*

Proof. It suffices to prove that, for every k , $U \in N_k\text{Open}(M)$ and (0) above, choices for (1) form a contractible space.

Note that the required condition implies $p_i^{-1}(s_i) \subset p_j^{-1}[s_j, s'_j]$ for $i \leq j$. Now the space of s'_k satisfying $p_i^{-1}(s_i) \subset p_k^{-1}[s_k, s'_k]$ for every $i \leq k$, is contractible. Moreover, once s'_j is chosen so we have

$$p_i^{-1}(s_i) \subset p_j^{-1}[s_j, s'_j]$$

for all $i \leq j$, then the space of s'_{j-1} satisfying $p_i^{-1}(s_i) \subset p_{j-1}^{-1}[s_{j-1}, s'_{j-1}]$ for every $i \leq j-1$, and $p_{j-1}^{-1}[s_{j-1}, s'_{j-1}] \subset p_j^{-1}[s_j, s'_j]$, is contractible. Moreover, once s'_j 's are all chosen, so these conditions are satisfied, then the space of t'_j 's satisfying the required condition is contractible. This completes the proof. \square

Before describing X_κ for $\kappa \geq 1$, let us construct a map $X_0 \rightarrow N_0 \text{laxcolim}_{\Delta^{\text{op}}} N_\bullet \mathcal{A}$ over $N_0 \Delta$, to be the simplicial level 0 of the desired map $X_\bullet \rightarrow N_\bullet \text{laxcolim}_{\Delta^{\text{op}}} N_\bullet \mathcal{A}$. For an integer $k \geq 0$, let

$$X_{0k} := \coprod_{U \in N_k \text{Open}(M)} X_U \quad (= X_0 \times_{N_0 \mathcal{X}} N_k \text{Open}(M)),$$

so $X_0 = \text{colim}_{[k] \in N_0 \Delta} X_{0k}$. We construct maps $X_{0k} \rightarrow N_k \mathcal{A}$ for all k , and define the desired map as the colimit (coproduct) of these maps over $[k] \in N_0 \Delta$. The map $X_{0k} \rightarrow N_k \mathcal{A}$ will be defined by induction on k as follows.

We first define a map $X_{00} \rightarrow N_0 \mathcal{A}$. Thus, suppose given $U \in \text{Open}(M)$ and a pair consisting of $p: \overline{U} \rightarrow I$ and $I' \subset I$ as above. Then define $p': \overline{U} \rightarrow I'$ as the composite of p with the map $I \twoheadrightarrow I'$ which collapses each of $[s, s']$ and $[t', t]$. Let j' denote the inclusion $I' - \{t'\} \hookrightarrow I'$. Then we associate to the point (U, p, I') of X_{00} the point $\int_{I'} j'_! j'^* p'_* A$ of $N_0 \mathcal{A}$. This is functorial on the groupoid X_{00} by the functoriality of the push-forward construction on the space of constructible maps (Remark 2.14), and the value is canonically equivalent to the compactly supported homology $\int_U^c A$. Note that the value for (U, p, I') can also be written as $\int_{I'} j'_! j'^* (p'|_{\overline{U}'})_* A$, where $\overline{U}' := p^{-1}[s, t'] \subset \overline{U}$.

Next, we construct a map $X_{01} \rightarrow N_1 \mathcal{A}$. Thus, consider a point of X_{01} given by a 1-simplex $U: U_0 \hookrightarrow U_1$ of $N_\bullet \text{Open}(M)$, and $(p, I') \in X_U$. Then we would like to construct a 1-simplex of $N_\bullet \mathcal{A}$ to be associated to it.

In order to do this, define p'_i to be the composite of p_i with the map $I_i \twoheadrightarrow I'_i$ collapsing each of $[s_i, s'_i]$ and $[t'_i, t_i]$, and define $\overline{U}'_i := p_i^{-1}[s_i, t'_i]$. Then $\overline{U}_1 = \overline{U}'_0 \cup_{p_0^{-1}(t'_0)} (\overline{U}_1 - U'_0)$, and the maps p'_0 and $p'_1|_{\overline{U}_1 - U'_0}$ glue together to define a map

$$p: \overline{U}_1 \longrightarrow I'_0 \cup I'_1 =: I,$$

where the intervals are glued by the relation $t'_0 = s'_1$.

Using this, we have the following. Let j_0 denote the inclusion $I'_0 - \{t'_0\} \hookrightarrow I'_0$, j_1 denote the inclusion $I - \{t'_1\} \hookrightarrow I$, and q denote the map $I \twoheadrightarrow I'_0$ collapsing I'_1 . Then the induced augmentation of $B := q_* j_1! j_1^* p'_* A$ induces a map $B \rightarrow j_0! j_0^* B = j_0! j_0^* p'_0 A$.

Integrating this over I'_0 , we obtain a map

$$\int_{I'_1} j_{1!} j_1^* p'_{1*} A = \int_I j_{1!} j_1^* p_* A \longrightarrow \int_{I'_0} j_{0!} j_0^* p'_{0*} A.$$

The source and the target here are models of the compactly supported homology associated to the composites $\overline{U}_i \xrightarrow{p_i} I_i \rightarrow [s_i, t'_i]$ for $i = 1, 0$ respectively, where the latter map collapses the subinterval $[t'_i, t_i]$ of I_i . The 1-simplex in \mathcal{A} we would like to associate is this map with 0-faces given by its source and target. This 1-simplex (as an object of the groupoid $N_1 \mathcal{A}$) is functorial on the groupoid X_{01} , since the push-forward of the augmentation map (together with the push-forward of the algebra) depends functorially on the space of constructible maps by Remark 2.14. Therefore, we have constructed a map $X_{01} \rightarrow N_1 \mathcal{A}$ as desired.

Next, we construct a map $X_{02} \rightarrow N_2 \mathcal{A}$. Thus, suppose given a 2-simplex $U: U_0 \hookrightarrow U_1 \hookrightarrow U_2$ of $N_\bullet \text{Open}(M)$, and $(p, I') \in X_U$. In order to construct a 2-simplex of $N_\bullet \mathcal{A}$ from these data, note that the previous constructions applied to 0- and 1-faces of (U, p, I') (by which we mean the faces of the nerve U equipped with the restrictions of (p, I') there) give what could be the boundary of a 2-simplex $[2] \rightarrow \mathcal{A}$. In order to fill inside of this by a 2-isomorphism to actually get a 2-simplex, define $p'_i: \overline{U}_i \rightarrow I'_i$ and $\overline{U}'_i \subset \overline{U}_i$ as before. Then the maps $p'_0: \overline{U}'_0 \rightarrow I'_0$, $p'_1: \overline{U}'_1 - U'_0 \rightarrow I'_1$, $p'_2: \overline{U}_2 - U'_1 \rightarrow I'_2$ glue together to define a map

$$p: \overline{U}_2 \longrightarrow I'_0 \cup I'_1 \cup I'_2 =: I,$$

where the union is taken under the relations $t'_i = s'_{i+1}$. Let $q_{ij}: I \rightarrow I_{ij} := I'_i \cup I'_j$ be the map collapsing the other interval, $q_j: I_{ij} \rightarrow I'_j$ be the map collapsing I'_i , $j_2: I - \{t'_2\} \hookrightarrow I$ be the inclusion, and $B := q_{01*} j_{2!} j_2^* p_* A$ on I_{01} . Then the 2-isomorphism we would like to find is between $\int_{I'_0}$ of the augmentation map $q_{0*} B \rightarrow j_{0!} j_0^* q_{0*} B$, and $\int_{I'_0}$ of the composite

$$q_{0*} B \xrightarrow{q_{0*} \varepsilon} q_{0*} j_{1!} j_1^* B \xrightarrow{\varepsilon} j_{0!} j_0^* q_{0*} j_{1!} j_1^* B = j_{0!} j_0^* q_{0*} B$$

of augmentation maps. We find an isomorphism between these, induced from the data of multiplicativity/monoidality of the augmentation map. Moreover, the 2-simplex we have thus constructed again depends functorially on X_{02} , since the push-forward of all data associated with augmentation maps are functorial on the space of constructible maps by Remark 2.14. Therefore, we have constructed a map $X_{02} \rightarrow N_2 \mathcal{A}$.

Inductively, let a point $(U, p, I') \in X_{0k}$ be given by $U \in N_k \text{Open}(M)$ and $(p, I') \in X_U$. Then the previous steps applied to the boundary faces of (U, p, I') give a diagram of the shape of $\partial \Delta^k$ in \mathcal{A} . We would like to fill this by a k -isomorphism to obtain a k -simplex $[k] \rightarrow \mathcal{A}$, which we can then associate to (U, p, I') .

In the case $k = 3$, the previous step of the induction implies that the 1-faces of this $\partial \Delta^3$ -shaped diagram are the 1-isomorphisms induced from the suitable instances of the augmentation map ε of A , and the 2-faces are the 2-isomorphisms induced from the appropriate instances of the 2-isomorphism of multiplicativity of the augmentation map ε . We get a 3-simplex in \mathcal{A} by filling the inside of this $\partial \Delta^3$ shape by the 3-isomorphism in \mathcal{A} induced from the 3-isomorphism of coherence of the multiplicativity of the augmentation map. We associate this 3-simplex of $N_\bullet \mathcal{A}$ to $(U, p, I') \in X_{03}$.

In the case $k = 4$, the faces of dimension up to 2 of the $\partial \Delta^4$ -shaped diagram are similar to those in the previous case, and the 3-faces are the instances of 3-simplices constructed in the previous step from the 3-isomorphisms of coherence of the multiplicativity of the augmentation map. We get a 4-simplex by filling inside

this $\partial\Delta^4$ shape by the 4-isomorphism induced from the 4-isomorphism here of the next level coherence of the multiplicativity of the augmentation map.

For a general k , we may assume by induction, that the previous steps are done by similarly taking the isomorphisms of dimension up to $k - 1$, all induced from the appropriate instances of the coherence isomorphisms. In particular, the simplices we obtain from the boundary faces of (U, p, I') , are made up of the isomorphisms induced from the coherence isomorphisms (up to dimension $k - 1$) of the multiplicativity of the augmentation map, and the shape of $\partial\Delta^k$ in \mathcal{A} is therefore made up of these simplices. Then, as promised by the inductive hypothesis, we associate to (U, p, I') the k -simplex $[k] \rightarrow \mathcal{A}$ obtained by filling inside this $\partial\Delta^k$ shape by the k -isomorphism in \mathcal{A} induced from the k -isomorphism here of the next level coherence of the multiplicativity of the augmentation map of A . This construction of a k -simplex is again functorial on X_{0k} . Therefore, we have constructed a map $X_{0k} \rightarrow N_k\mathcal{A}$ in the way required by the next inductive hypothesis. This completes the induction on k , and therefore the construction of a map $X_0 \rightarrow N_0 \text{laxcolim}_{\Delta^{\text{op}}} N_{\bullet}\mathcal{A}$ over $N_0\Delta$.

Let us next construct the space X_1 . We define it as the coproduct over $N_1\mathcal{X}$ of certain spaces, each of which will turn out contractible. First, note that a point of $N_1\mathcal{X}$ is specified by integers $k, \ell \geq 0$, a map $\varphi: [k] \rightarrow [\ell]$ in Δ , and $U \in N_\ell \text{Open}(M)$. Given a point of $N_1\mathcal{X}$ specified by these data, we let the space naturally formed by the following be the component of X_1 lying over it.

- (0) A point $(p, I') \in X_U$.
 - (1) For every $i \in [k]$, a subinterval $J_i = [u_i, v_i]$ of $I'_{\varphi i}$ such that $v_i \leq u_{i+1}$ whenever $\varphi i = \varphi(i+1)$.
 - (2) A map in the fundamental groupoid of the space
- $$(3.2) \quad \text{Map}_{/[\ell]}(\varphi, I') := \prod_{j \in [\ell]} \text{Map}_{\leq}(\varphi^{-1}j, I'_j)$$

(where Map_{\leq} stands for the space of order preserving maps), from $v = (v_i)_{i \in [k]}$ to $\varphi^* t' = (t'_{\varphi i})_{i \in [k]}$, where we are identifying a point of $\text{Map}_{/[\ell]}(\varphi, I')$ in general with an increasing sequence $x = (x_i)_{i \in [k]}$ of points in $\bigcup_{j \in [\ell]} I'_j$ such that $x_i \in I'_{\varphi i}$.

Note, as we have claimed, that the space is contractible, so the projection $X_1 \rightarrow N_1\mathcal{X}$ is an equivalence.

More generally, for an integer $\kappa \geq 2$, we let X_κ be the coproduct over $N_\kappa\mathcal{X}$ of the following (again, contractible) spaces. Namely, let a point of $N_\kappa\mathcal{X}$ be specified by a κ -nerve

$$(3.3) \quad \varphi: [k_\kappa] \xrightarrow{\varphi_\kappa} \cdots \xrightarrow{\varphi_1} [k_0]$$

and $U \in N_{k_0} \text{Open}(M)$, where we consider $[\kappa]$ as $\{\kappa \rightarrow \cdots \rightarrow 0\}$ for notational convenience. Then we take the natural space formed by the following, for the component of X_κ to lie over the specified point of $N_\kappa\mathcal{X}$.

- (0) A point $(p, J^0) \in X_U$.
- (1) For every $\iota \in [\kappa] - \{0\}$, a family $J^\iota = (J^\iota_i)_{i \in [k_\iota]}$ of subintervals $J^\iota_i = [u^\iota_i, v^\iota_i]$ of $J^{\iota-1}_{\varphi_\iota i}$, satisfying $v^\iota_i \leq u^\iota_{i+1}$ if $\varphi_\iota i = \varphi_\iota(i+1)$.
- (2) For every $\iota \in [\kappa] - \{0\}$, a map in the fundamental groupoid of the space $\text{Map}_{/[k_{\iota-1}]}(\varphi_\iota, J^{\iota-1})$ (see (3.2) above), from v^ι to $\varphi_\iota^*(v^{\iota-1})$.

Note that this space is contractible as claimed, so the projection $X_\kappa \rightarrow N_\kappa\mathcal{X}$ is an equivalence.

It follows that there uniquely exists a pair consisting of a simplicial structure on X_\bullet and a simplicial structure on the level-wise map $X_\bullet \rightarrow N_\bullet \mathcal{A}$ given by the projections. We will want to use a more concrete description of this.

In order to get a desired description of the simplicial structure, for $\iota \in [\kappa]$, let ψ_ι denote the composite $\varphi_1 \cdots \varphi_\iota : [k_\iota] \rightarrow [k_0]$. Then note that the pair $(\psi_\iota^* p, J^\iota)$ determines a point of $X_{\psi_\iota^* U}$. Now suppose given a map $f : [\lambda] \rightarrow [\kappa]$ in Δ . We would like to define a map $f^* : X_\kappa \rightarrow X_\lambda$. In order to do this, suppose given a point of X_κ over the point $(\varphi, U) \in N_\kappa \mathcal{A}$, specified as above. Then we let f^* associate to this point a point of X_λ lying over $f^*(\varphi, U) = (f^* \varphi, (f^* \varphi)_0^* U) \in N_\lambda \mathcal{A}$, specified by

- (0) the point $(\psi_{f(0)}^* p, J^{f(0)}) \in X_{\psi_{f(0)}^* U} = X_{(f^* \varphi)_0^* U}$,
- (1) $f^* J = (J^{f(\iota)})_{\iota \in [\lambda] - \{0\}}$, and
- (2) for every $\iota \in [\lambda] - \{0\}$, the map $v^{f(\iota)} \rightarrow (f^* \varphi)_\iota^* (v^{f(\iota-1)})$ in the fundamental groupoid of $\text{Map}_{/[k_{f(\iota-1)}]}((f^* \varphi)_\iota, J^{f(\iota-1)})$, obtained by composing the paths determining the point of X_κ .

This extends to a map $X_\kappa \rightarrow X_\lambda$ by functoriality, and we define f^* as this map. By the associativity of composition of maps in the fundamental groupoids, this construction defines a functoriality on Δ , and we have thus described a simplicial structure of X_\bullet , lying over the simplicial structure of $N_\bullet \mathcal{A}$.

Let us finally extend the map $X_0 \rightarrow N_0 \text{laxcolim}_{\Delta^{\text{op}}} N_\bullet \mathcal{A}$ over $N_0 \Delta$ we have constructed, to a full simplicial map over $N_\bullet \Delta$.

For this purpose, suppose given the following partial data towards a 1-simplex of X_\bullet . The data we consider are a 1-simplex $(\varphi, U) \in N_1 \mathcal{A}$ ($\varphi : [k_1] \rightarrow [k_0]$), which corresponds to a component of X_1 , a family $p = (p_j)_{j \in [k_0]}$ of constructible maps $p_j : \overline{U}_j \rightarrow I_j$ as before, $J^0 = (J_j^0)_{j \in [k_0]}$ ($J_j^0 = [u_j^0, v_j^0] \subset I_j$) such that $(p, J^0) \in X_U$. Then we construct a map $\text{Map}_{/[k_0]}(\varphi, J^0) \rightarrow N_{k_1} \mathcal{A}$ as follows. Namely, let a point of $\text{Map}_{/[k_0]}(\varphi, J^0)$ be represented by an increasing sequence $x = (x_i)_{i \in [k_1]}$ of $\bigcup_{[k_0]} J^0$. Then we construct a k_1 -simplex $[k_1] \rightarrow \mathcal{A}$ from this data by making the following modifications to the construction of the map $X_{0k_1} \rightarrow N_{k_1} \mathcal{A}$ we have done before.

To be precise on the comparison with the previous construction, what we shall construct is a k_1 -simplex of $N_\bullet \mathcal{A}$ which specialises to the k_1 -simplex associated to $(\varphi^* U, \varphi^* p, J^1) \in X_{0k_1}$ (denote it by $\int_{\varphi^* U, \varphi^* p, J^1}^c A$) if $x = v^1$ for a family $J^1 = (J_i^1)_{i \in [k]}$ of subintervals $J_i^1 = [u_i^1, v_i^1] \subset J_{\varphi_i}^0$ satisfying the conditions we have described in the definition of the space X_1 , so $(\varphi^* p, J^1) \in X_{\varphi^* U}$. The modification will be made to the construction of $\int_{\varphi^* U, \varphi^* p, J^1}^c A$ from the data $(\varphi^* U, \varphi^* p, J^1) \in X_{0k_1}$. Its description as follows.

Firstly, by denoting $p_{\varphi_i}^{-1}[s_{\varphi_i}, v_i^1]$ by \overline{V}_i , the construction of $\int_{\varphi^* U, \varphi^* p, J^1}^c A$ used the constructible map

$$\bigcup_{i \in [k_1]} q_i|_{\overline{V}_i - V_{i-1}} : \overline{V}_{k_1} \longrightarrow \bigcup_{i \in [k_1]} J_i^1,$$

where $q_i : \overline{U}_{\varphi_i} \xrightarrow{p_{\varphi_i}} I_{\varphi_i} \rightarrow J_i^1$. In the construction for x , we instead use

$$\bigcup_{j \in \varphi[k_1]} p'_j|_{\overline{U}'_{j'} - U'_{j'}} : \overline{U}_{\varphi(k_0)} \longrightarrow \bigcup_{j \in \varphi[k_1]} J_j^0,$$

where j' denotes the element of $\varphi[k_1] \subset [k_0]$, previous to j . Moreover, whenever we push-forward an algebra to J_i^1 in the original construction, we instead push the corresponding algebra forward to $[s'_{\varphi_i}, x_i]$. The rest of the construction will be unchanged. The construction is functorial on $\text{Map}_{/[k_0]}(\varphi, J^0)$.

Note that, for the point $x = \varphi^*(v^0) \in \text{Map}_{/[k_0]}(\varphi, J^0)$, we obtain the k_1 -simplex $\varphi^* \int_{U,p,J^0}^c A$. In particular, if we are given a 1-simplex of X_\bullet in the component for $(\varphi, U) \in N_1 \mathcal{X}$, specified by $p, J = (J^\iota)_{\iota \in [1]}$ as above, and a map $\alpha: v^1 \xrightarrow{\sim} \varphi^*(v^0)$ in $\text{Map}_{/[k_0]}(\varphi, J^0)$, then α induces an equivalence $\int_{\varphi^* U, \varphi^* p, J^1}^c A \xrightarrow{\sim} \varphi^* \int_{U,p,J^0}^c A$, or equivalently, a (Cartesian) map $\int_{\varphi^* U, \varphi^* p, J^1}^c A \rightarrow \int_{U,p,J^0}^c A$ in $\text{laxcolim}_{\Delta^{\text{op}}} N_\bullet \mathcal{A}$, covering the map φ in Δ .

Using this, for every fixed $\kappa \geq 1$, we construct the map $X_\kappa \rightarrow N_\kappa \text{laxcolim}_{\Delta^{\text{op}}} N_\bullet \mathcal{A}$ over $N_\kappa \Delta$ as follows. Namely, let a point of X_κ in the component for $(\varphi, U) \in N_\kappa \mathcal{X}$ be specified by (p, J, α) , $J = (J^\iota)_{\iota \in [\kappa]}$, $\alpha = (\alpha^\iota)_{\iota \in [\kappa] - \{0\}}$, $\alpha^\iota: v^\iota \xrightarrow{\sim} \varphi_\iota^*(v^{\iota-1})$, as before. Then applying the above construction to the 1-faces of this κ -simplex connecting the adjacent vertices, we obtain a sequence of maps

$$(3.4) \quad \int_{\psi_\kappa^* U, \psi_\kappa^* p, J^\kappa}^c A \longrightarrow \cdots \longrightarrow \int_{\psi_\iota^* U, \psi_\iota^* p, J^\iota}^c A \longrightarrow \cdots \longrightarrow \int_{U,p,J^0}^c A$$

in $\text{laxcolim}_{\Delta^{\text{op}}} N_\bullet \mathcal{A}$. Then using composition of maps in $\text{laxcolim}_{\Delta^{\text{op}}} N_\bullet \mathcal{A}$, we obtain a κ -simplex $[\kappa] \rightarrow \text{laxcolim}_{\Delta^{\text{op}}} N_\bullet \mathcal{A}$. Since the construction is functorial on X_κ , we obtain a desired map.

Moreover, the collection over $[\kappa] \in \Delta$ of these maps $X_\kappa \rightarrow N_\kappa \text{laxcolim}_{\Delta^{\text{op}}} N_\bullet \mathcal{A}$ over $N_\kappa \Delta$, has a functoriality in $[\kappa] \in \Delta$, coming from our construction of the simplicial structure of X_\bullet by compositions (and their associativity) of maps in the fundamental groupoids of spaces of increasing sequences we used, and the construction of the sequence (3.4) by functoriality on these groupoids. This completes the construction of a functoriality of $U \mapsto \int_U^c A$ on $\text{Open}(M)^{\text{op}}$.

3.2. Symmetric monoidality. Next, we would like to give the compactly supported homology functor $\int_?^c A: \text{Open}(M)^{\text{op}} \rightarrow \mathcal{A}$ a natural symmetric monoidal structure. Recall from Section 1.1 that, for us, this means extending the functor to a map of the functors $\text{Fin}_* \rightarrow \text{Cat}$ (i.e., “pre- Γ -categories”). The pre- Γ -category to be the target here is the underlying functor $S_+ \mapsto \mathcal{A}^S$ of the symmetric monoidal category \mathcal{A} . The source is the functor on Fin_* defined by $S_+ \mapsto \text{Open}^{(S)}(M)^{\text{op}}$, where $\text{Open}^{(S)}(M)$ is the full subposet of $\text{Open}(M)^S$ consisting of families $U = (U_s)_{s \in S}$ of pairwise disjoint open submanifolds of M , indexed by S .

Thus, it suffices to show that our construction of the functor $\text{Open}(M)^{\text{op}} \rightarrow \mathcal{A}$ extends to $\text{Open}^{(S)}(M)^{\text{op}} \rightarrow \mathcal{A}^S$ in a way functorial in S_+ . This can be done by defining $\mathcal{X}^{(S)} := \text{laxcolim}_{\Delta^{\text{op}}} N_\bullet \text{Open}^{(S)}(M)^{\text{op}}$, and concretely constructing simplicial spaces $X^{(S)}$ and maps

$$N_\bullet \mathcal{X}^{(S)} \xleftarrow{\sim} X_\bullet^{(S)} \longrightarrow N_\bullet \text{laxcolim}_{\Delta^{\text{op}}} (N_\bullet \mathcal{A})^S = N_\bullet \text{laxcolim}_{\Delta^{\text{op}}} N_\bullet \mathcal{A}^S$$

over $N_\bullet \Delta$, functorially in S_+ , extending the previous construction from the case where S is one point.

We define the simplicial space $X_\bullet^{(S)}$ so the κ -th space is the coproduct over $N_\kappa \mathcal{X}^{(S)}$ of the following (contractible) spaces. Namely, let $(\varphi, U) \in N_\kappa \mathcal{X}^{(S)}$, where φ is a κ -nerve (3.3) in Δ , and U is a k_0 -simplex $U_0 \hookrightarrow \cdots \hookrightarrow U_{k_0}$ of $N_\bullet \text{Open}^{(S)}(M)$, where $U_i = (U_{is})_{s \in S}$ is a family of disjoint open submanifolds of M , and each U_{*s} is a k_0 -simplex of $N_\bullet \text{Open}(M)$. Then we let the component of $X_\kappa^{(S)}$ corresponding to (φ, U) be the same as the component of X_κ corresponding to $(\varphi, \bigsqcup_S U)$ of $N_\kappa \mathcal{X}$. As we have observed before, these spaces are contractible, so the projection $X_\kappa^{(S)} \rightarrow N_\kappa \mathcal{X}^{(S)}$ is an equivalence. Note that a concrete description of the unique lift of the simplicial structure of $N_\bullet \mathcal{X}^{(S)}$ to $X_\bullet^{(S)}$, is obtained by pulling back the concrete description of the lift of the simplicial structure along $X_\bullet \rightarrow N_\bullet \mathcal{X}$.

Moreover, since the map $X_{\bullet}^{(S)} \rightarrow N_{\bullet}\mathcal{X}^{(S)}$ is an equivalence, the functoriality in S_+ of $N_{\bullet}\mathcal{X}^{(S)}$ lifts for $X_{\bullet}^{(S)}$ uniquely. We will want to use the following concrete description of a lift. Namely, suppose given a map $f: S_+ \rightarrow T_+$ in Fin_* . Then for a simplex U of $N_{\bullet}\text{Open}^{(S)}(M)$, we have $\bigsqcup_T f_! U = \bigsqcup_{f^{-1}T} U$ for the Γ -structure map $f_!$ of $\text{Open}(M)$. Therefore, if (φ, U) is a simplex of $N_{\kappa}\mathcal{X}^{(S)}$, then the data for specifying a point in the corresponding component of $X_{\kappa}^{(S)}$, namely, in the component of X_{κ} corresponding to $(\varphi, \bigsqcup_S U) \in N_{\kappa}\mathcal{X}$, can be restricted to $\bigsqcup_T f_! U$, to specify a point in the component of $X_{\kappa}^{(T)}$ corresponding to $f_!(\varphi, U) = (\varphi, f_! U) \in N_{\kappa}\mathcal{X}^{(T)}$. This defines a map $X_{\kappa}^{(S)} \rightarrow X_{\kappa}^{(T)}$, which is functorial in κ by our concrete description of the simplicial structures. By taking the resulting map $N_{\kappa}\mathcal{X}^{(S)} \rightarrow N_{\kappa}\mathcal{X}^{(T)}$ to be the structure map $f_!$, we obtain a concrete description of a Γ -structure of X_{\bullet} , lifting that of $N_{\bullet}\mathcal{X}$.

In order to construct a map $X_{\bullet}^{(S)} \rightarrow N_{\bullet}\text{laxcolim}_{\Delta^{\text{op}}}(N_{\bullet}\mathcal{A})^S$ over $N_{\bullet}\Delta$, note that $N_{\kappa}\text{laxcolim}_{\Delta^{\text{op}}}(N_{\bullet}\mathcal{A})^S$ is the collection of the components of $N_{\kappa}(\text{laxcolim}_{\Delta^{\text{op}}} N_{\bullet}\mathcal{A})^S$ lying over the diagonal of $N_{\kappa}\Delta^S$. Thus, we shall construct a map $X_{\bullet}^{(S)} \rightarrow (N_{\bullet}\text{laxcolim}_{\Delta^{\text{op}}} N_{\bullet}\mathcal{A})^S$ landing in these components. It suffices to describe the component for each $s \in S$ of this map. We let it be the composite

$$X_{\bullet}^{(S)} \xrightarrow{\text{pr}_s} X_{\bullet} \xrightarrow{f_s^c A} N_{\bullet}\text{laxcolim}_{\Delta^{\text{op}}} N_{\bullet}\mathcal{A},$$

where pr_s is the map corresponding to the inclusion $\{s\}_+ \hookrightarrow S_+$ in the Γ -structure of X_{\bullet} , which has been concretely described above. This gives a map $X_{\bullet}^{(S)} \rightarrow N_{\bullet}\text{laxcolim}_{\Delta^{\text{op}}}(N_{\bullet}\mathcal{A})^S$ over $N_{\bullet}\Delta$, as desired.

Finally, we want functoriality of these maps in S_+ . However, using the concrete description of the Γ -structure of X_{\bullet} , this results immediately from the symmetric monoidality of A and the augmentation map of A , and our assumption on the monoidal structure of \mathcal{A} (see Section 1.2).

4. KOSZUL DUALITY FOR FACTORISATION ALGEBRAS

4.0. The Koszul dual of a factorisation algebra. We have seen in the previous section that compactly supported homology with coefficients in an augmented locally constant factorisation algebra A , is contravariantly functorial, and symmetric monoidal, in the open submanifolds. Let us denote this symmetric monoidal functor by A^+ . Namely, for an open submanifold U of the manifold M on which A is defined, we denote $A^+(U) := \int_U^c A$.

We can restrict this coalgebra on $\text{Open}(M)$ to $\text{Disj}(M)$, and consider it as a coalgebra on the multicategory $\text{Disk}(M)$. Let us denote this coalgebra by $A^!$.

Definition 4.0. Let \mathcal{A} be a symmetric monoidal category which is closed under sifted colimits, and whose monoidal multiplication preserves sifted colimits. Let A be an augmented locally constant factorisation algebra on a manifold M , taking values in \mathcal{A} . Then the **Koszul dual** of A is defined as the augmented locally constant coalgebra $A^!$ on $\text{Disk}(M)$, taking values in \mathcal{A} .

Even though $A^!$ came from a functor A^+ , A^+ may not in general satisfy any reasonable descent property. However, if the symmetric monoidal structure of \mathcal{A} behaves well with both sifted colimits *and* sifted limits, then the results of [10, Section 2], in particular, Theorem 2.11, can be applied in \mathcal{A}^{op} . In particular, there is a universal way to extend $A^!$ to a functor on $\text{Open}(M)$ satisfying factorising descent. In this case, one may expect A^+ to be close to the functor extended from $A^!$ by descent.

One of the cases is where \mathcal{A} is the category Space^{op} of the opposite spaces with the coCartesian symmetric monoidal structure. In this case, A^+ satisfies factorising

descent as often as one may expect. This is indeed Lurie’s “nonabelian Poincaré duality” theorem [9] (which is closely related to earlier results of Segal [15], McDuff [13] and Salvatore [14]) as we shall discuss in Section 4.1.

Thus, compactly supported factorisation homology gives a context generalising the context for this theorem. As another case where the monoidal structure behaves well with sifted colimits and sifted limits, we shall analyse in Section 4.1, the case where \mathcal{A} is Space with the Cartesian symmetric monoidal structure. We will find in this case that the theorem of Lurie’s type (which in fact is equivalent to Lurie’s theorem) admits a refinement which does not seem to exist in the opposite context (see Remark 4.8).

In the later sections, we shall describe a result in which the symmetric monoidal structure is *not* required to behave well with sifted limits, but the behaviour of the functor A^+ can still be nice thanks to some additional structure on the target category.

For the remainder of this section, we shall see some simple examples of the Koszul dual coalgebras. More specifically, we shall see instances of the following, easy consequence of the constructions. Let us denote the category of augmented locally constant factorisation algebras by $\text{Alg}_{M,*}(\mathcal{A})$, and the category of augmented locally constant coalgebras on $\text{Disk}(M)$ by $\text{Coalg}_{M,*}(\mathcal{A})$.

Proposition 4.1. *Let \mathcal{A} and \mathcal{B} be symmetric monoidal category which are closed under sifted colimits, and whose symmetric monoidal multiplication preserves sifted colimits. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a symmetric monoidal functor which preserves sifted colimits. Then, the canonical map filling the square*

$$\begin{array}{ccc} \text{Alg}_{M,*}(\mathcal{A}) & \xrightarrow{(\)^!} & \text{Coalg}_{M,*}(\mathcal{A}) \\ F \downarrow & & \downarrow F \\ \text{Alg}_{M,*}(\mathcal{B}) & \xrightarrow{(\)^!} & \text{Coalg}_{M,*}(\mathcal{B}) \end{array}$$

is an equivalence.

One example of a functor F as in the proposition is given by the Koszul duality functor. Namely, let \mathcal{A} be a symmetric monoidal category which is closed under sifted colimits, and whose symmetric monoidal multiplication preserves sifted colimits. Then we can apply the proposition to the functor $(\)^!: \text{Alg}_{N,*}(\mathcal{A}) \rightarrow \text{Coalg}_{N,*}(\mathcal{A})$, where N is any manifold without boundary.

Let us apply the proposition on another manifold M . Then we obtain that the canonical map filling the square

$$(4.2) \quad \begin{array}{ccc} \text{Alg}_{M,*}(\text{Alg}_{N,*}) & \xrightarrow{(\)^!} & \text{Coalg}_{M,*}(\text{Alg}_{N,*}) \\ (\)^! \downarrow & & \downarrow (\)^! \\ \text{Alg}_{M,*}(\text{Coalg}_{N,*}) & \xrightarrow{(\)^!} & \text{Coalg}_{M,*}(\text{Coalg}_{N,*}), \end{array}$$

where we have dropped \mathcal{A} from our notation, is an equivalence.

In fact, it is immediate to see that the diagonal map here is canonically equivalent to the composite

$$\text{Alg}_{M \times N,*} \xrightarrow{(\)^!} \text{Coalg}_{M \times N,*} \xrightarrow{\text{restriction}} \text{Coalg}_{M,*}(\text{Coalg}_{N,*}),$$

through the identification $\text{Alg}_{M \times N,*} \xrightarrow{\sim} \text{Alg}_{M,*}(\text{Alg}_{N,*})$ by the restriction functor following from [10, Theorem 3.14].

For example, the Koszul dual on \mathbb{R}^n can be understood as the result of iteration of taking the Koszul dual on \mathbb{R}^1 . The definition in terms of the compactly supported homology gives a coordinate-free description of the same thing.

We shall give a few more examples of symmetric monoidal functors to which Proposition 4.1 applies. (We will be very far from being comprehensive (and from being most general). Our purpose is to discuss just a few of many examples for illustration.)

Example 4.3. Fix a base field (in the usual discrete sense) of characteristic 0, and let (Lie, \oplus) and (Mod, \oplus) denote the symmetric monoidal category of Lie algebras and of modules respectively, over the base field, with the symmetric monoidal structure given by the direct sum operations. (We may consider dg Lie algebras and dg modules (chain complexes).) When we do not specify the symmetric monoidal structure in the notation as above, let us understand we are taking the symmetric monoidal structures given by the tensor product (over the base field).

Then any of the symmetric monoidal functors appearing in the following commutative diagram preserves sifted colimits.

$$\begin{array}{ccccccc}
 & & (\text{Mod}, \oplus) & \xrightarrow[\sim]{\Sigma} & (\text{Mod}, \oplus) & & \\
 & \swarrow = & \downarrow & & \downarrow \text{Sym} & & \\
 (\text{Mod}, \oplus) & \xleftarrow{\text{forget}} & (\text{Lie}, \oplus) & \xrightarrow{C_\bullet} & \text{Coalg}_{\text{Com},*} & \xrightarrow{\text{forget}} & \text{Coalg}_{\text{Com}} \\
 & & \downarrow U & & \downarrow \text{forget} & & \downarrow \text{forget} \\
 & & \text{Alg}_{E_n,*} & \xrightarrow{(\)^!} & \text{Coalg}_{E_n,*} & \xrightarrow{\text{forget}} & \text{Coalg}_{E_n} \xrightarrow{\text{forget}} \text{Mod},
 \end{array}$$

where $\Sigma = (\)[1]$ is the suspension functor, $(\text{Mod}, \oplus) \hookrightarrow (\text{Lie}, \oplus)$ is the inclusion of Abelian Lie algebras, Com stands for “commutative” ($= E_\infty$), C_\bullet is the Lie algebra homology functor, and U is the enveloping E_n -algebra functor (where $n \neq \infty$).

Proposition 4.1 gives a result on comparison of the Koszul dual coalgebras through any of these functors.

Corollary 4.4. *The functor $(\)^! : \text{Alg}_M(\text{Lie}, \oplus) \rightarrow \text{Sh}_M(\text{Lie})$ is an equivalence. Compactly supported homology and cohomology satisfy descent.*

Proof. $(\)^! : \text{Alg}_M(\text{Mod}, \oplus) \rightarrow \text{Coalg}_M(\text{Mod}, \oplus)$ is the Verdier functor $\text{Cosh}_M(\text{Mod}) \rightarrow \text{Sh}_M(\text{Mod})$, and is easily seen to be an equivalence by looking at what it does at the level of stalks.

The result follows since the functor $\text{Lie} \rightarrow \text{Mod}$ reflects equivalences. \square

The functor

$$\text{Sh}_M(\text{Lie}) \xrightarrow[\sim]{(\)^!} \text{Alg}_M(\text{Lie}, \oplus) \xrightarrow{C_\bullet} \text{Alg}_M(\text{Mod})$$

is particularly interesting since by the work of Costello and Gwilliam [1], [6], for a particular sheaf \mathfrak{g} of Lie algebras over $\mathbb{C}[[\hbar]]$ (the Heisenberg Lie algebras), the factorisation algebra $C_\bullet(\mathfrak{g}^!)$ is the factorisation algebra of observables of the (deformation) quantisation of a free classical field theory, in the framework of [1].

Proposition 4.1 applies to this functor. Note that if M is a Euclidean space, then the category of sheaves of Lie algebras is just the category of Lie algebras, since our sheaves are assumed to be locally constant.

The compactly supported homology of the factorisation algebra $C_\bullet(\mathfrak{g}^!)$ is easy to describe. Namely, we have

$$\int_U^c C_\bullet(\mathfrak{g}^!) = C_\bullet \int_U^c \mathfrak{g}^! = C_\bullet(\mathfrak{g}(U)).$$

More on this will be discussed in Section 4.4.

4.1. Descent properties of compactly supported factorisation homology.

In this section, we examine the Koszul duality in particular situations where the monoidal structure preserves both sifted colimits and sifted limits variable-wise.

As a first example, we observe that Lurie’s “nonabelian Poincaré duality” theorem [9] we have introduced in Section 0.0 is a theorem about the Koszul duality for factorisation algebras. Indeed, there, we have stated the theorem in terms of a functor E^+ obtained from a sheaf E of spaces, by taking compactly supported cohomology. The sheaf E may be a locally constant sheaf of spaces in the infinity 1-categorical sense, in order for the theorem to make sense, and to be true. Such E can be identified exactly with a locally constant factorisation algebra A taking values in Space^{op} . Then the prealgebra E^+ in Space gets identified with the precoalgebra A^+ in Space^{op} we have defined in Section 4.0.

Thus, Lurie’s theorem is along the line of discussions we have made after Definition 4.0 in Section 4.0.

See [9] for the relation of this to the classical ‘Abelian’ or ‘stable’ Poincaré duality theorem. In the classical context, the role of the Koszul duality is played by the Verdier duality.

Another interesting point mentioned in [9] is that at a point of M , the stalk of $A^!$ is the n -fold based loop space of the stalk of E , and the structure of a factorisation algebra of $A^!$ is extending the structure of an E_n -algebra of the n -fold loop space. Namely, the Koszul duality construction in the current context is globalising the looping functor in the context of the classical theory of iterated loop spaces. We shall next consider a globalisation of the *delooping* functor.

We go to the opposite context, and consider the case where the algebra A takes values in Space , the category of spaces with the Cartesian symmetric monoidal structure.

The following is a version of non-Abelian duality theorem in this context. It can be proved more or less similarly to Lurie’s theorem. However, this proposition can be also deduced from Lurie’s theorem, and vice versa.

Proposition 4.5. *If every stalk of A is group-like as an E_1 -algebra, then A^+ is a locally constant factorisation algebra in Space^{op} . In particular, the map $\int_M^c A \rightarrow \int_M A^!$ is an equivalence.*

In the present context, the formal part of the proof of Gromov’s h-principle applies, and we obtain the following. In the opposite context, there does not seem to be a similar theorem (at least in an interesting way). See Remark 4.8.

Theorem 4.6. *Let A be a locally constant factorisation algebra of spaces on a manifold M . Then the canonical map $\int_M^c A \rightarrow \int_M A^! (\simeq \Gamma(M, A^!)$, derived sections) of spaces is an equivalence if no connected component of M is a closed manifold (i.e., if M is “open”).*

Proof. Note that the association $U \mapsto \Gamma(U, A^!)$ is the universal locally constant sheaf associated to the locally constant presheaf A^+ , and the map $\int_M^c A \rightarrow \int_M A^!$ is the map on the global sections of the universal map.

Take a handle body decomposition of \overline{M} involving no handle of index $n := \dim M$, and for this decomposition, $A^! : U \mapsto \Gamma(U, A^!)$ satisfies excision for every handle attachment.

We first prove that excision for attachment of a handle of index smaller than n is satisfied by A^+ as well. More generally, suppose W is an open submanifold of M which as a manifold by itself, is given as the interior of a compact manifold \overline{W} ,

and let $\{U, V\}$ be a cover of \overline{W} by two open submanifolds (possibly with boundary) which has a diffeomorphism $U \cap V \xrightarrow{\sim} N \times \mathbb{R}^1$ for a $(n-1)$ -dimensional manifold N , compact with boundary. In the case of handle body attachment (so V , say is the attached handle, and \overline{W} is the result of attachment), N is of the form $S^{i-1} \times \overline{\mathbb{R}}^{n-i}$, where i is the index of the handle.

Let us denote $\mathring{U} = U \cap W$, $\partial U = U \cap \partial \overline{W} (\subsetneq \partial \overline{U})$, and similarly for V and N . Then $A^+(\mathring{U} \cap \mathring{V}) = A^+(\mathring{N} \times \mathbb{R}^1)$ is an E_1 -coalgebra, and let us assume that our choice of orientation of \mathbb{R}^1 makes $A^+(\mathring{U})$ a right, and $A^+(\mathring{V})$ a left module respectively, over this E_1 -coalgebra. Then we want to show that the restriction map

$$A^+(W) \longrightarrow A^+(\mathring{U}) \square_{A^+(\mathring{U} \cap \mathring{V})} A^+(\mathring{V}),$$

where the target denotes the cotensor product, is an equivalence.

Recall that

$$A^+(W) = A(W) \otimes_{A(\partial \overline{W})} \mathbf{1}.$$

Our idea is to apply the excision property of A to $A(W)$ and $A(\partial \overline{W})$ in a compatible way. That is, using the decomposition of \overline{W} into U and V , and its restriction to the boundary (or rather, a collar of $\partial \overline{W}$), we obtain identifications

$$A(W) = A(\mathring{U}) \otimes_{A(\mathring{U} \cap \mathring{V})} A(\mathring{V})$$

and

$$A(\partial \overline{W}) = A(\partial U) \otimes_{A(\partial U \cap \partial V)} A(\partial V)$$

which are compatible with the actions at boundary. Therefore, by denoting by G the E_1 -algebra $A(\mathring{U} \cap \mathring{V}) \otimes_{A(\partial U \cap \partial V)} \mathbf{1}$, we obtain that

$$A^+(W) = K \otimes_G L,$$

where K is the right G -module $A(\mathring{U}) \otimes_{A(\partial U)} \mathbf{1}$, and L the similar left G -module corresponding to V . This follows since the difference between the objects in question is difference in the order in which to realise a bisimplicial object.

However, in terms of these algebra and modules, we see that $A^+(\mathring{U}) \otimes A^+(\mathring{V}) = K \otimes_G \mathbf{1} \otimes_G L$, and $A^+(\mathring{U} \cap \mathring{V}) = \mathbf{1} \otimes_G \mathbf{1}$, and by inspecting the actions, we find that the assertion of excision is that the canonical map

$$K \otimes_G G \otimes_G L \longrightarrow (K \otimes_G \mathbf{1}) \square_{\mathbf{1} \otimes_G \mathbf{1}} (\mathbf{1} \otimes_G L),$$

defined by algebra is an equivalence. We state this as a lemma below, and the proof of the lemma will complete the proof of the excision property of A^+ for handle body attachment. The proof of lemma will use the assumption $i \leq n-1$.

The proof can be now completed similarly to the proof of Theorem 2.11, by induction on the number of handles in the decomposition of \overline{M} we have been considering. Namely, since both A^+ and $A^!$ satisfy excision for handle body attachment of index smaller than $\dim M$, by induction, it suffices to prove that the value of these functors agree on open submanifolds of M diffeomorphic to either a disk or $\partial D^i \times D^{n-i+1}$, where $i \leq n-1$.

The case of a disk is by the definition of $A^!$. The latter case follows from the excision for the handle body attachment of index smaller than n , since $\partial D^i \times \overline{D}^{n-i+1}$ can be obtained by attaching a handle of index $i-1$ to a disk. \square

Let us state and prove the lemma which was promised in one of the steps in the proof. To recall the notation, G is an E_1 -algebra of spaces, and K is a left, and L is a right, G -module respectively, both of spaces. In the previous proof, we were in the situation where $G = A(\mathring{N} \times \mathbb{R}^1) \otimes_{A(\partial N \times \mathbb{R}^1)} \mathbf{1}$, where $N = S^i \times \overline{\mathbb{R}}^{n-i}$,

and $n - i \geq 1$, and A was a locally constant factorisation algebra on $\mathring{N} \times \mathbb{R}^1$. In particular, the underlying space of G was connected.

For the following lemma, we only need to assume that G is group-like, namely, the monoid $\pi_0(G)$ is in fact a group. Since this assumption is satisfied for a connected G , the proof of the following lemma closes the unfinished step of the previous proof.

Lemma 4.7. *The canonical map*

$$K \otimes_G L \longrightarrow (K \otimes_G \mathbf{1}) \times_{G^\dagger} (\mathbf{1} \otimes_G L)$$

is an equivalence for every K and L if it is so for $K = G$ and $L = G$, namely if G is group-like.

Proof. Consider the maps as a map over G^\dagger . Then the induced map on the fibres over the unique (up to homotopy) point of $G^\dagger = BG$ can be identified with the identity of $K \times L$. \square

Remark 4.8. In the previous context where the target category \mathcal{A} is Space^{op} with the coCartesian symmetric monoidal structure, there does not seem to be a result corresponding to Theorem 4.6, in an interesting way.

In fact, if A is an arbitrary algebra in \mathcal{A} , namely, a locally constant sheaf of spaces, then the map $\tilde{A} \rightarrow A$ from the stalk-wise n -connective cover (where n is the dimension of our manifold) induces an equivalence on the Koszul dual. Therefore, for any open U , we have $\int_U A^\dagger = \Gamma_c(U, \tilde{A})$.

However, it happens only rarely that the map $\tilde{A} \rightarrow A$ induces an equivalence on the space of compactly supported sections.

4.2. Poincaré duality for complete factorisation algebras. In this section, let \mathcal{A} be a symmetric monoidal complete soundly filtered stable category with uniformly bounded sequential limits, as defined in [11, Sections 2, 3, Definition 4.3].

Let us first recall the following.

Definition 4.9 ([11, Definition 4.16]). An augmented E_n -algebra A in \mathcal{A} is said to be **positive** if its augmentation ideal belongs to $\mathcal{A}_{\geq 1}$.

An augmented E_n -coalgebra C in \mathcal{A} is said to be **copositive** if there is a uniform bound ω for loops [11, Definition 2.42] in \mathcal{A} such that the augmentation ideal J of C belongs to $\mathcal{A}_{\geq 1-n\omega}$.

Let M be a manifold (without boundary). We shall prove a version of the non-Abelian Poincaré duality theorem in \mathcal{A} , for factorisation algebras which is *positive* in the following sense.

Definition 4.10. An augmented factorisation algebra A on M is said to be **positive** if the stalk A_x at every point $x \in M$ is positive as an E_n -algebra, where $n = \dim M$.

Theorem 4.11. *Let A be a positive augmented locally constant factorisation algebra on M , valued in \mathcal{A} as above. Then, A^+ , defined by compactly supported factorisation homology (see Section 4.0), satisfies excision.*

For the proof we use the following facts established in [11].

For an associative coalgebra C , let us denote by $-\square_C-$ the (co-)tensor product over C .

Proposition 4.12 ([11, Proposition 4.7]). *Let A be a positive augmented associative algebra, and C a copositive augmented associative coalgebra, both in \mathcal{A} . Assume A is positive, and C is copositive (Definition 4.9). Let K be a right A -module, L an A - C -bimodule, and let X be a left C -module, all bounded below.*

Then the canonical map

$$K \otimes_A (L \square_C X) \longrightarrow (K \otimes_A L) \square_C X$$

is an equivalence.

Theorem 4.13 ([11, Theorem 4.10]). *Let A be a positive augmented associative algebra in \mathcal{A} , and K be a right A -module which is bounded below. Then the canonical map $K \rightarrow (K \otimes_A \mathbf{1}) \square_{\mathbf{1} \otimes_A \mathbf{1}} \mathbf{1}$ is an equivalence (of A -modules).*

Proof of Theorem 4.11. As in the proof of Theorem 4.6, suppose that a situation for excision is given as follows.

Namely, suppose W is an open submanifold of M which as a manifold by itself, is given as the interior of a compact manifold \overline{W} , and let $\{U, V\}$ be a cover of \overline{W} by two open submanifolds (possibly with boundary) which has a diffeomorphism $U \cap V \xrightarrow{\sim} N \times \mathbb{R}^1$ for an $(n-1)$ -dimensional manifold N , compact with boundary.

Let us denote $\mathring{U} = U \cap W$, $\partial U = U \cap \partial \overline{W} (\subsetneq \partial \mathring{U})$, and similarly for V and N . Then $A^+(\mathring{U} \cap \mathring{V}) = A^+(\mathring{N} \times \mathbb{R}^1)$ is an E_1 -coalgebra, and let us assume that our choice of orientation of \mathbb{R}^1 makes $A^+(\mathring{U})$ right, and $A^+(\mathring{V})$ left modules over this E_1 -coalgebra.

Then, as before, we have an E_1 -algebra $B := A(\mathring{U} \cap \mathring{V}) \otimes_{A(\partial U \cap \partial V)} \mathbf{1}$ (denoted by G before), a right B -module $K := A(\mathring{U}) \otimes_{A(\partial U)} \mathbf{1}$, and L the similar left B -module corresponding to V , and the excision in question can be stated as that the canonical map

$$K \otimes_B B \otimes_B L \longrightarrow (K \otimes_B \mathbf{1}) \square_{\mathbf{1} \otimes_B \mathbf{1}} (\mathbf{1} \otimes_B L)$$

defined by algebra, where the target denotes the cotensor product, is an equivalence.

In order to prove this, it suffices to note that B is a positive augmented algebra. Indeed, it follows that the map above is an equivalence by the above results of [11]. \square

Remark 4.14. Note that even if we do not assume that the monoidal operations preserve sifted colimits, the proof works if the prealgebra satisfies excision.

Theorem has the following interesting consequence. A version of this was obtained earlier by Francis ([2, 3]). To give a context, let us recall that if A is an E_n -algebra, then A is an n -dualisable object of the Morita $(n+1)$ -category $\text{Alg}_n(\mathcal{A})$ (see Lurie [8]). A concrete description of the associated topological field theory can be outlined as follows [8]. A defines a locally constant factorisation algebra on every framed n -dimensional manifold which can be considered to be ‘(globally) constant at A ’ in a sense. In particular, for $k \leq n$, if M is a $(n-k)$ -dimensional manifold equipped with a framing of $M \times \mathbb{R}^k$ (so M might be a $(n-k)$ -morphism in the n -dimensional framed cobordism category), then we obtain an E_k -algebra by pushing forward the constant factorisation algebra on $M \times \mathbb{R}^k$, along the projection $M \times \mathbb{R}^k \rightarrow \mathbb{R}^k$. Let us denote this E_k -algebra by $\int_M A$. If M is indeed a $(n-k)$ -morphism in the n -dimensional framed cobordism category, then $\int_M A$ interact with the factorisation homology of A over the manifolds appearing as its sources and targets of all codimensions, in a certain specific way to make it a $(n-k)$ -morphism in $\text{Alg}_n(\mathcal{A})$. Excision then implies that the association $M \mapsto \int_M A$ is functorial with respect to the compositions in the cobordism category, so this gives an n -dimensional fully extended framed topological field theory. The value of this theory for an n -framed point p is indeed the E_n -algebra A .

For \mathcal{A} a complete soundly filtered stable with uniformly bounded sequential limits, we have shown in [11, Section 4.3] that the Koszul duality has a Morita theoretic functoriality. Namely, we have constructed a positive augmented coalgebraic version $\text{Coalg}_n^+(\mathcal{A})$ of the higher Morita category, and have shown that the construction

of the Koszul duals gives a symmetric monoidal functor $\text{Alg}_n^+(\mathcal{A}) \rightarrow \text{Coalg}_n^+(\mathcal{A})$, which is an equivalence [11, Theorem 4.22]. (The source here is the positive part of the Morita $(n+1)$ -category of augmented algebras.) If A is augmented and positive, then the field theory in $\text{Alg}_n^+(\mathcal{A})$ associated to A corresponds via this functor, to a theory in $\text{Coalg}_n^+(\mathcal{A})$ associated to $A^!$.

On the other hand, a consequence of Theorem 4.11 is that there is an n -dimensional fully extended topological field theory in $\text{Coalg}_n^+(\mathcal{A})$ which associates to an $(n-k)$ -morphism M in the framed cobordism category, the compactly supported homology $\int_{M \times \mathbb{R}^k}^c A$ (with suitable algebraic structure) of the factorisation algebra on $M \times \mathbb{R}^k$, “constant at A ”. (To be accurate, $\int_{M \times \mathbb{R}^k}^c A$ is a slightly more general than what we have considered, in that M is compact with boundary and other higher codimensional corners. However, since we are dealing only with constant coefficients, there is almost no difficulty added in establishing their basic behaviour.) The value for the n -framed point p of this theory is the E_n -coalgebra $\int_{\mathbb{R}^n}^c A = A^!$. (See (4.2).)

The cobordism hypothesis implies that there is a unique equivalence between these two theories in $\text{Coalg}_n^+(\mathcal{A})$ which fixes the common value for the n -framed point. Such an equivalence can be concretely seen by writing

$$\int_{M \times \mathbb{R}^k}^c A = \int_{\mathbb{R}^k}^c \int_M A = \left(\int_M A \right)^!,$$

where $\int_M A$ for all M are understood to be equipped with the algebraic structures to make them morphisms in $\text{Alg}_n^+(\mathcal{A})$. In fact, the excision situations we need to consider to check the functoriality of $\int_{M \times \mathbb{R}^k}^c A$ in the cobordism category, all reduce to the situations we considered in checking the Morita functoriality of the Koszul duality construction [11, Theorem 4.22], and we have used the identical arguments in both situations.

As another consequence of the Poincaré duality theorem 4.11, we obtain an equivalence of categories from the Koszul duality on a manifold, generalising the equivalence on \mathbb{R}^n (Theorem 0.1). This shows that the Koszul duality for factorisation algebras is a non-Abelian extension of the Verdier duality.

Let us define the suitable category of augmented coalgebras. Let $\text{Open}_1^{\text{loc}}(M)$ denote the following multicategory. We define it by first defining its category of colours, and then giving it a structure of a multicategory.

First let Man denote the discrete category whose object is a “manifold without boundary” in our convention stated in Section 1.2, and where maps are “open embeddings” as defined at the same place. Man is a symmetric monoidal category under disjoint union.

Let Man^{loc} denote the following, non-discrete version of Man . Namely, its objects are the same as Man , but we take the space of morphisms to be the *space* of open embeddings. Let Man_1 denote the full subposet of Man consisting of manifolds with exactly one connected component.

We define the category of colours of $\text{Open}_1^{\text{loc}}(M)$ as $\text{Man}_1 \times_{\text{Man}^{\text{loc}}} \text{Man}_{/M}^{\text{loc}}$.

This (as any category) is the category of colours of a multicategory where a multimap is simply a family of maps. The structure of a multicategory we consider on $\text{Man}_1 \times_{\text{Man}^{\text{loc}}} \text{Man}_{/M}^{\text{loc}}$ to define $\text{Open}_1^{\text{loc}}(M)$, is a restriction of this structure of a multicategory, where we require the family of maps in Man_1 specified as a part of the data of a family of maps in $\text{Man}_1 \times_{\text{Man}^{\text{loc}}} \text{Man}_{/M}^{\text{loc}}$, to be pairwise disjoint (over the interiors). Namely, given a finite set S and a family $U = (U_s)_{s \in S}$ of objects, and an object V , a multimap $U \rightarrow V$ is by definition an open embedding $\coprod_S U \hookrightarrow V$ together with for each $s \in S$, a path in the space $\text{Emb}(U_s, M)$, from the defining embedding $U_s \hookrightarrow M$ to the composite $U_s \hookrightarrow V \hookrightarrow M$.

Let us denote by $\text{Coalg}_M(\mathcal{A})_+$ the category of *copositive* augmented coalgebras on $\text{Open}_1^{\text{loc}}(M)$, valued in \mathcal{A} which satisfies excision, where by **copositive**, we mean that every stalk of the augmented coalgebra is copositive as an augmented E_n -coalgebra (Definition 4.9).

Theorem 4.15. *Let \mathcal{A} be a symmetric monoidal complete soundly filtered stable category with uniformly bounded sequential limits. Assume that the monoidal operations preserves sifted colimits (variable-wise). Then the functor*

$$(\)^+ : \text{Alg}_M(\mathcal{A})_+ \longrightarrow \text{Coalg}_M(\mathcal{A})_+$$

is an equivalence.

Proof. The inverse is given by taking compactly supported ‘co’-homology. Namely, if $C \in \text{Coalg}_M(\mathcal{A})_+$ (namely, is a *copositive* augmented coalgebra), then the pre-algebra C^+ defined by $C^+(U) = \int_U^c C$ (the definition of Section 3 applied in \mathcal{A}^{op} using the copositivity) satisfies excision by the proof similar to the proof of Theorem 4.11, and hence is a (positive augmented) factorisation algebra.

One checks that this functor is indeed inverse to the given functor, by looking at what these functors do to the stalks of algebras and coalgebras. At the level of stalks, the functor is the Koszul duality construction of E_n -algebras and coalgebras, done by iteration of E_1 -Koszul duality constructions. The result now follows from Theorem 0.1, which is obtained by iterating the E_1 -case of it [11, Sections 4.0, 4.1]. \square

In particular, the functor $M \mapsto \text{Coalg}_M(\mathcal{A})_+$ satisfies descent for (effectively) factorising l-nice bases as in [10, Theorem 2.26]. If we had known this descent property first, then Theorem could have been deduced from the local case Theorem 0.1. A direct proof of the descent property might involve some interesting extension of our methods developed in [10, Section 2].

Remark 4.16. If the assumption of preservation of sifted colimits by the monoidal operations is not satisfied, then a positive augmented algebra on $\text{Open}_1^{\text{loc}}(M)$ satisfying excision still seems to be a meaningful notion. Theorem remains true for such objects. See also Remark 4.14 and [11, Remark 4.27].

4.3. Example of a positive factorisation algebra. In this section, we show that a factorisation algebra in any reasonable symmetric monoidal stable category canonically gives rise to a positive factorisation algebra in the symmetric monoidal category of filtered objects there [11, Section 3.2]. It will follow that the Poincaré duality theorem applies to the completion of this positive filtered factorisation algebra (Corollary 4.25 below).

Let \mathcal{A} be a symmetric monoidal stable category. Assume \mathcal{A} has all sequential limits in it. Let A be an augmented factorisation algebra in \mathcal{A} on a manifold M . We would like to define a filtered factorisation algebra $F_\bullet A$. (Recall from Section 1.2 that we are assuming that the monoidal multiplication preserves colimits variable-wise.)

Let us first introduce some notations. Let Fin denote the category of finite sets. Let Fin^{nu} denote the category of finite sets, with surjections as maps. For an integer $r \geq 0$, let $\text{Fin}_{\geq r}$ denote the full subcategory of Fin^{nu} consisting of objects T with at least r elements.

Given any category \mathcal{C} equipped with a functor $\mathcal{C} \rightarrow \text{Fin}$, define

$$\mathcal{C}_{\geq r} := \mathcal{C} \times_{\text{Fin}} \text{Fin}_{\geq r}$$

whenever the choice of the functor to Fin is understood. This is a full subcategory of

$$\mathcal{C}^{\text{nu}} := \mathcal{C}_{\geq 0} = \mathcal{C} \times_{\text{Fin}} \text{Fin}^{\text{nu}}.$$

For example, for any manifold M , from the functor $\pi_0: \text{Disj}(M) \rightarrow \text{Fin}$ which associates to a disjoint union of disks, the finite set of its components, we obtain a poset $\text{Disj}^{\text{nu}}(M) := \text{Disj}(M)^{\text{nu}}$ and its full subposets $\text{Disj}_{\geq r}(M) := \text{Disj}(M)_{\geq r}$.

Example 4.17. $\text{Disj}_{\geq 0}(U) = \text{Disj}^{\text{nu}}(U) = \text{Disj}_{\geq 1}(U) \sqcup \{\emptyset\}$.

If U is empty, then $\text{Disj}_{\geq \bullet}(U)$ is the unit ‘filtered’ category, namely $\text{Disj}_{\geq 0}(U) = *$, and $\text{Disj}_{\geq r}(U) = \emptyset$ for $r \geq 1$.

We shall often represent an object of $\text{Disj}^{\text{nu}}(M)$ as a pair (T, D) , where $T \in \text{Fin}^{\text{nu}}$, and $D = (D_t)_{t \in T}$ is a family of disjoint disks in M , indexed by T .

Let us start a construction of $F_{\bullet}A$ for an augmented factorisation algebra A .

Let us denote by \tilde{A} the reduced version of A , which can be considered as a symmetric monoidal functor on $\text{Disj}^{\text{nu}}(M)$. Namely, given a pair $(T, D) \in \text{Disj}^{\text{nu}}(M)$, \tilde{A} associates to it the object

$$\tilde{A}(D) := \bigotimes_{t \in T} I(D_t),$$

where $I := \text{Fibre}[\varepsilon: A \rightarrow \mathbf{1}]$ (the section-wise fibre).

For an open submanifold U of M , we define

$$F_r A(U) = \text{colim}_{\text{Disj}_{\geq r}(U)} \tilde{A}.$$

In other words, if \mathbb{Z} denotes the integers made into a category by their order as

$$\cdots \longleftarrow r \longleftarrow r+1 \longleftarrow \cdots,$$

then the functor $\mathbb{Z} \ni r \mapsto F_r A(U)$ is the left Kan extension of \tilde{A} along the functor $\text{Disj}^{\text{nu}}(U) \xrightarrow{\pi_0} \text{Fin}^{\text{nu}} \xrightarrow{\text{card}} \mathbb{Z}$, where ‘card’ takes the cardinality of finite sets.

Example 4.18. It follows from Example 4.17 that the values of the augmentation ideal $IF_{\bullet}A = \text{Fibre}[\varepsilon: F_{\bullet}A \rightarrow \mathbf{1}]$ are positive (i.e., ‘ ≥ 1 ’) in the filtration.

If U is empty, then $F_{\bullet}A(U) = \mathbf{1}$.

If U is a disk, then $\text{Disj}_{\geq 1}(U)$ has a maximal element (a terminal object), so $F_1 A(U) = \tilde{A}(U)$ and $F_0 A(U) = A(U)$.

We would like to prove that $F_{\bullet}A$ defines a locally constant factorisation algebra of filtered objects.

Lemma 4.19. *Let U, V be manifolds. Then the functor*

$$\text{Disj}_{\geq r}(U \amalg V) \longleftarrow \text{laxcolim}_{i+j \geq r} \text{Disj}_{\geq i}(U) \times \text{Disj}_{\geq j}(V)$$

given by taking disjoint unions, has a left adjoint. In particular, it is cofinal.

Proof. The left adjoint is given as follows. Let (T, D) be an element of $\text{Disj}_{\geq r}(U \amalg V)$. Then T, D can be written uniquely as

$$T = T' \amalg T'', \quad D = D' \amalg D''$$

where D' (resp. D'') is a collection of disks in U (resp. V), indexed by T' (resp. T'').

From these, we obtain an element (T', D') (resp. (T'', D'')) of $\text{Disj}_{\geq \sharp T'}(U)$ (resp. $\text{Disj}_{\geq \sharp T''}(V)$). We map (T, D) to $(T', D') \times (T'', D'')$ in the lax colimit. Note that $\sharp T' + \sharp T'' = \sharp T \geq r$, so this is well-defined.

In order to verify that the object $(T', D') \times (T'', D'')$ of the colimit satisfies the required universal property, let another object of the colimit, $(T'_1, D'_1) \times (T''_1, D''_1)$ be given, where $(T'_1, D'_1) \in \text{Disj}_{\geq i_1}(U)$, $(T''_1, D''_1) \in \text{Disj}_{\geq j_1}(V)$, for i_1, j_1 such that $i_1 \leq \sharp T'_1$, $j_1 \leq \sharp T''_1$ (and $i_1 + j_1 \geq r$). Suppose furthermore that we have a map $(T, D) \rightarrow (T_1, D_1)$ in $\text{Disj}_{\geq r}(U \amalg V)$, where $T_1 = T'_1 \amalg T''_1$ and $D_1 = D'_1 \amalg D''_1$.

Recall that such a map was a surjective map $f: T \rightarrow T_1$ such that for every $t \in T$, $D_t \subset D_{1,f(t)}$. It follows that f decomposes uniquely as a map $f' \amalg f'': T' \amalg T'' \rightarrow T'_1 \amalg T''_1$, where f', f'' are surjections.

In particular, we have $\sharp T' \geq \sharp T'_1 \geq i_1$ and $\sharp T'' \geq j_1$, and the universal property follows immediately. \square

Remark 4.20. The unit for the adjunction is an equivalence, while the counit gets inverted in the (non-lax) colimit. It follows that the map

$$\mathrm{Disj}_{\geq r}(U \amalg V) \longleftarrow \operatorname{colim}_{i+j \geq r} \mathrm{Disj}_{\geq i}(U) \times \mathrm{Disj}_{\geq j}(V)$$

is an equivalence.

Lemma 4.21. *Let M be a manifold. Then the map*

$$\operatorname{laxcolim}_{U \in \mathrm{Disj}(M)} \mathrm{Disj}_{\geq r}(U) \longrightarrow \mathrm{Disj}_{\geq r}(M)$$

has a left adjoint. In particular, it is cofinal.

Proof. This is obvious. \square

Remark 4.22. A similar remark as the remark to Lemma 4.19 applies to this lemma as well.

For a manifold M , define $D^{\mathrm{nu}}(M) := D(M)^{\mathrm{nu}}$, and $D_{\geq r}(M) := D(M)_{\geq r}$, with respect to the functor $\pi_0: D(M) \rightarrow \mathbf{Fin}$ which takes the connected components of a disjoint union of disks. We represent an object of $D^{\mathrm{nu}}(M)$ as a pair (T, D) as before.

Lemma 4.23. *For every r , the functor $\mathrm{Disj}_{\geq r}(M) \rightarrow D_{\geq r}(M)$ is cofinal. Moreover, $D_{\geq r}(M)$ is sifted.*

Proof. It suffices to prove that for a finite cover $p: \widetilde{M} \twoheadrightarrow M$, the functor $\mathrm{Disj}_{\geq r}(M) \rightarrow D_{\geq r}(\widetilde{M})$ given by taking the inverse images under p , is cofinal. Indeed [9, Section 5.3.2], the first statement is a special case. The second is equivalent to that the functor $\Delta: D_{\geq r}(M) \rightarrow D_{\geq r}(\widetilde{M})$ is cofinal for non-empty \widetilde{M} . This follows if the composite $\mathrm{Disj}_{\geq r}(M) \rightarrow D_{\geq r}(M) \xrightarrow{\Delta} D_{\geq r}(\widetilde{M})$ is cofinal.

The proof of this upgraded statement is similar to the proof of [9, Proposition 5.3.2.13 (1)]. Namely, we apply Joyal's generalisation of Quillen's theorem A [7] to reduce it to the following. Namely, let (T_0, D_0) be an object of $D_{\geq r}(\widetilde{M})$, and denote the defining embedding $\coprod_{T_0} D_0 \hookrightarrow \widetilde{M}$ by i . Then it suffices to prove that the category

$$\operatorname{laxcolim}_{(T,D) \in \mathrm{Disj}_{\geq r}(M)} \coprod_{f: T_0 \twoheadrightarrow T} \mathrm{Fibre} \left[\coprod_{t \in T} \mathrm{Emb} \left(\coprod_{f^{-1}(t)} D_0, p^{-1} D_t \right) \longrightarrow \mathrm{Emb} \left(\coprod_{T_0} D_0, \widetilde{M} \right) \right]$$

has a contractible classifying space, where the fibre is that over i .

It follows from homotopy theory (see Remark 2.4) that the classifying space is equivalent to the space

$$\mathrm{Fibre} \left[\operatorname{colim}_{(T,D) \in \mathrm{Disj}_{\geq r}(M)} \coprod_{f: T_0 \twoheadrightarrow T} \coprod_{t \in T} \mathrm{Emb} \left(\coprod_{f^{-1}(t)} D_0, p^{-1} D_t \right) \longrightarrow \mathrm{Emb} \left(\coprod_{T_0} D_0, \widetilde{M} \right) \right].$$

Furthermore, we obtain that it suffices to prove that the map

$$\operatorname{colim}_{(T,D) \in \mathrm{Disj}_{\geq r}(M)} \coprod_{f: T_0 \twoheadrightarrow T} \coprod_{t \in T} \mathrm{Conf}(f^{-1}(t), p^{-1} D_t) \longrightarrow \mathrm{Conf}(T_0, \widetilde{M})$$

is an equivalence. See the proof of Lemma 2.3.

The equivalence follows from applying the generalised Seifert–van Kampen theorem to the following open cover of $\text{Conf}(T_0, \widetilde{M})$. The cover is indexed by the category $\text{Disj}_{\geq r}(M)_{T_0/}$, and is given by the functor which associates to (T, D) with a map $f: T_0 \twoheadrightarrow T$, the open subset $\prod_{t \in T} \text{Conf}(f^{-1}(t), p^{-1}D_t)$ of $\text{Conf}(T_0, \widetilde{M})$.

It is immediate to see that this cover satisfies the assumption for the generalised Seifert–van Kampen theorem. \square

We conclude as follows.

Proposition 4.24. *Let \mathcal{A} be a symmetric monoidal stable category whose monoidal multiplication preserves colimits variable-wise. Assume that \mathcal{A} is closed under all sequential limits.*

Let A be an augmented factorisation algebra in \mathcal{A} on a manifold M . Then the augmented filtered prealgebra $F_\bullet A$ is a positive locally constant filtered factorisation algebra.

Proof. The functor $F_\bullet A$ on $\text{Open}(M)$ is symmetric monoidal by Lemma 4.19, Lemma 4.23, and the assumption that the monoidal operations preserve colimits variablewise.

It is locally constant by Lemma 4.23 (cofinality).

It is the left Kan extension from its restriction to $\text{Disj}(M)$, by Lemma 4.21.

It is positive by Example 4.18. \square

Corollary 4.25. *The completion $\widehat{F}_\bullet A$ of $F_\bullet A$ is a positive locally constant factorisation algebra taking values in the complete filtered stable category of the complete filtered objects of \mathcal{A} . In particular, the Poincaré duality theorem 4.11 applies to $\widehat{F}_\bullet A$.*

Proof. This is a consequence of Proposition and the general discussion in [11, Sections 3.1, 3.2] of monoidal filtration and filtered objects. \square

4.4. The dual theorems. Let \mathcal{A} be a stable category which is given a filtration, and is closed under sequential colimits. Then $\mathcal{B} := \mathcal{A}^{\text{op}}$ is a filtered stable category by defining $\mathcal{B}_{\geq r} = (\mathcal{A}^{\leq -r})^{\text{op}}$, where $\mathcal{A}^{\leq s} := \mathcal{A}^{< s+1}$.

If suspensions and sequential colimits in \mathcal{A} are bounded with respect to the filtration of \mathcal{A} , then the arguments we have made in the previous sections can be applied to \mathcal{B} .

Definition 4.26. Let \mathcal{A} be a stable category, and let a filtration and a monoidal structure \otimes (which is exact in each variable) be given on \mathcal{A} . We say that an integer p is an **upper bound** for the monoidal structure if $\mathbf{1}$ belongs $\mathcal{A}^{\leq 0}$, and for every integers r, s , the monoidal operation $\otimes: \mathcal{A}^2 \rightarrow \mathcal{A}$ takes the full subcategory $\mathcal{A}^{\leq r} \times \mathcal{A}^{\leq s}$ of the source to the full subcategory $\mathcal{A}^{\leq r+s+p}$ of the target.

A monoidal structure is said to be **bounded above** if it has an upper bound.

Proof of the following is similar to the proof of Proposition 4.12.

Proposition 4.27. *Let \mathcal{A} be a stable category which is closed under sequential colimits. Suppose given a filtration and a monoidal structure on \mathcal{A} , and assume that \mathcal{A}^{op} is complete with respect to the filtration. Let*

- d be an upper bound for sequential colimits in \mathcal{A} ,
- ω be an upper bound for suspensions in \mathcal{A} ,
- p be an upper bound for the monoidal structure.

*Let A be an augmented associative algebra, and C an augmented associative coalgebra, both in \mathcal{A} . Assume that the augmentation ideal of C belongs to $\mathcal{A}^{< -p}$. Assume A is **conegative** in the sense that its augmentation ideal belongs to $\mathcal{A}^{< -\omega - p} \cap \mathcal{A}^{< -p}$.*

Let K be a right A -module, L an A - C -bimodule, and let X be a left C -module, all bounded above.

Then the canonical map

$$K \otimes_A (L \square_C X) \longrightarrow (K \otimes_A L) \square_C X$$

is an equivalence.

It follows (if the monoidal operations preserves colimits variable-wise) that the Poincaré duality theorem similar to Theorem 4.11 holds for conegative factorisation algebras. (See the proof of Theorem 4.11.)

Example 4.28. Let \mathcal{A} be a stable category, and assume \mathcal{A} has all sequential colimits. Then the filtration on the category of filtered objects of \mathcal{A} from [11, Section 3.2] makes its opposite category a *complete* filtered stable category. All colimits which exist in \mathcal{A} , is bounded by 0 in the category of filtered objects.

Moreover, if \mathcal{A} is symmetric monoidal by operations which are exact variable-wise, then the monoidal structure on the filtered objects, described in [11, Section 3.2], is bounded above by 0.

Example 4.29. Let \mathfrak{g} be a dg Lie-algebra over a field (in the usual discrete sense) of characteristic 0. Then the Chevalley–Eilenberg complex $C_\bullet \mathfrak{g} = (\text{Sym}^*(\Sigma \mathfrak{g}), d)$ (where $\Sigma = ()[1]$ is the suspension functor, and the differential d is the sum of the internal differential from \mathfrak{g} and the Chevalley–Eilenberg differential) can be refined to give a filtered chain complex

$$\cdots \longrightarrow \mathbf{0} \longrightarrow \cdots \longrightarrow \mathbf{0} = F_1 C_\bullet \mathfrak{g} \longrightarrow F_0 C_\bullet \mathfrak{g} \longrightarrow \cdots \longrightarrow F_{-r} C_\bullet \mathfrak{g} \longrightarrow \cdots,$$

where $F_{-r} C_\bullet \mathfrak{g} := (\text{Sym}^{\leq r}(\Sigma \mathfrak{g}), d)$, so $C_\bullet \mathfrak{g} = \text{colim}_{r \rightarrow \infty} F_{-r} C_\bullet \mathfrak{g}$.

$F_* C_\bullet$ takes a quasi-isomorphism to quasi-isomorphism, so induces a functor between the infinity 1-categories where quasi-isomorphisms are inverted. This is since homological algebra implies that the homotopy cofibre $F_{-r} C_\bullet / F_{-r+1} C_\bullet$ is given by the quotient in the strict/discrete sense by the subcomplex, and preserves quasi-isomorphisms (since the symmetric group has vanishing higher homology for coefficients over our field). Note that the Chevalley–Eilenberg differential vanishes on the quotient.

Moreover, the functor preserves sifted homotopy colimits since the layers (or the associated graded) as above do.

Note furthermore that $F_* C_\bullet$ is obviously a lax symmetric monoidal functor.

Lemma 4.30. *The lax symmetric monoidal structure of $F_* C_\bullet$ is in fact a genuine symmetric monoidal structure.*

Proof. Let $\mathfrak{g}, \mathfrak{h}$ be dg Lie algebras. We would like to show that the canonical map

$$F_* C_\bullet \mathfrak{g} \otimes F_* C_\bullet \mathfrak{h} \longrightarrow F_* C_\bullet (\mathfrak{g} \oplus \mathfrak{h})$$

is a quasi-isomorphism.

It suffices to show that the map induces equivalence on all layers. Thus, let $r \geq 0$ be an integer. Then the $(-r)$ -th layer of the target is $\text{Sym}^r(\Sigma \mathfrak{g} \oplus \Sigma \mathfrak{h})$, as has been seen already.

The $(-r)$ -th layer of the source can be seen to be the direct sum over $i \geq 0$, $j \geq 0$ such that $i + j = r$, of the cofibre of the map

$$\text{colim}_{\substack{k \leq i, \ell \leq j \\ k + \ell \leq r-1}} F_{-k} C_\bullet \mathfrak{g} \otimes F_{-\ell} C_\bullet \mathfrak{h} \longrightarrow F_{-i} C_\bullet \mathfrak{g} \otimes F_{-j} C_\bullet \mathfrak{h}.$$

However, by inductive use of homological algebra, this cofibre can be seen to be $\text{Sym}^i(\Sigma \mathfrak{g}) \otimes \text{Sym}^j(\Sigma \mathfrak{h})$.

The result follows immediately. \square

It follows that if \mathfrak{g} is a locally constant sheaf of dg Lie algebras on a manifold M , then $F_*C_\bullet(\mathfrak{g}^!)$ is an (augmented) filtered dg factorisation algebra. Since it is negative, the Poincaré duality theorem of this section applies.

Theorem 4.31. *Let \mathfrak{g} be a locally constant sheaf of dg Lie algebras over a field of characteristic 0, on a manifold M . Then the (negative) filtered dg precoalgebra $F_*C_\bullet(\mathfrak{g})$ on M satisfies excision.*

Proof. It suffices to show $F_*C_\bullet(\mathfrak{g}) = F_*C_\bullet(\mathfrak{g}^!)+$. However, this can be done in the same way as the computation of $C_\bullet(\mathfrak{g}^!)+$ in Section 4.0. \square

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